On space efficiency of algorithms working on structural decompositions of graphs

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Dynamic programming on treewidth

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- We can do $c^{\mathbf{n}}$ time and poly space...
- Non-trivial space-efficient algos exist for other problems...

Time and space needed for algo on treewidth

3-COLORING:

- $2^{\mathcal{O}(\mathtt{tw})}\mathrm{poly}$ time and $2^{\mathcal{O}(\mathtt{tw})}\mathrm{poly}$ space by dynamic prog. ✓
- $2^{\mathcal{O}(\mathtt{tw}\cdot\log n)}$ time and $\mathtt{tw}\cdot\log n$ space by Div&Conq
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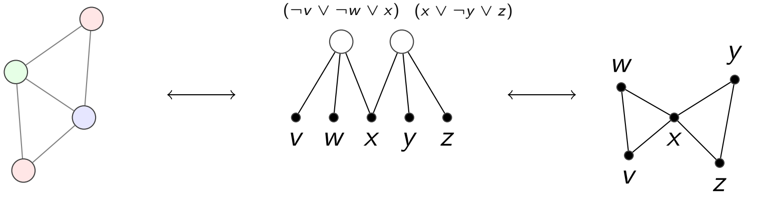
Equivalence of problems

Standard reductions preserve treewidth linearly,
so it's the same question for SAT, etc.

3-Coloring	k-SAT for `tw` of incidence graph	k-SAT or SAT for `tw` of primal graph



Completeness

- Fix a function like $s(n) = \log^5 n$.
- pw-3Col[s] – the problem 3-COLORING with a path decomposition of width at most $s(n)$ on input.

Thm

pw-3Col[s] is complete for $\mathbf{N}[\underbrace{\text{poly}}_{\textit{time}}, \underbrace{s}_{\textit{space}}]$

– problems solvable by **N**ondeterministic Turing machines with $\text{poly}(n)$ time and $\mathcal{O}(s(n))$ space.

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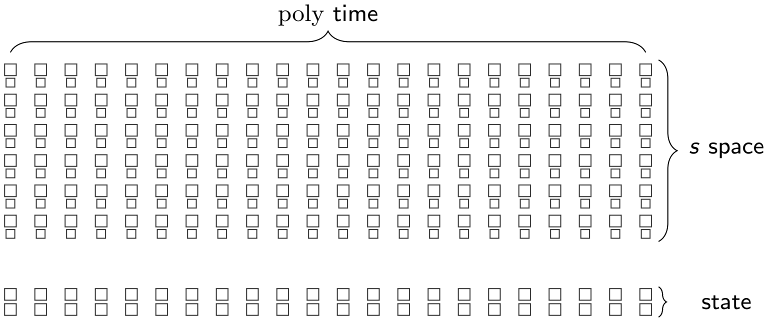

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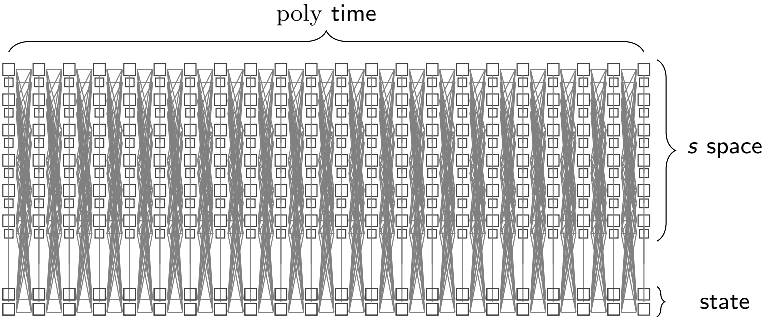

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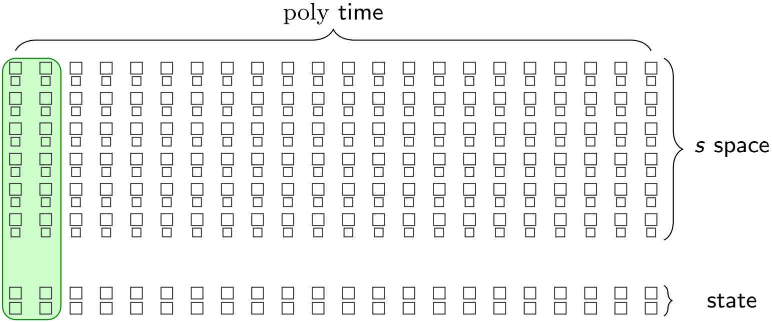

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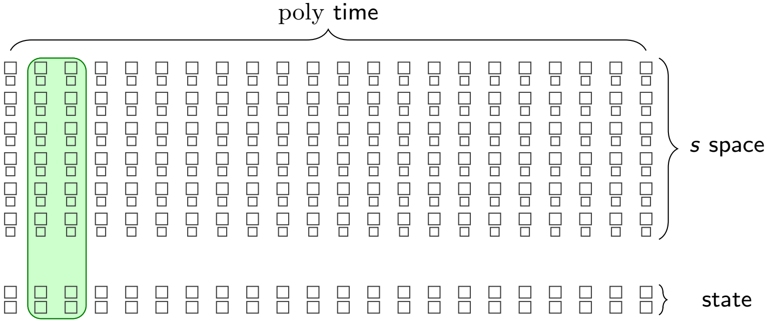

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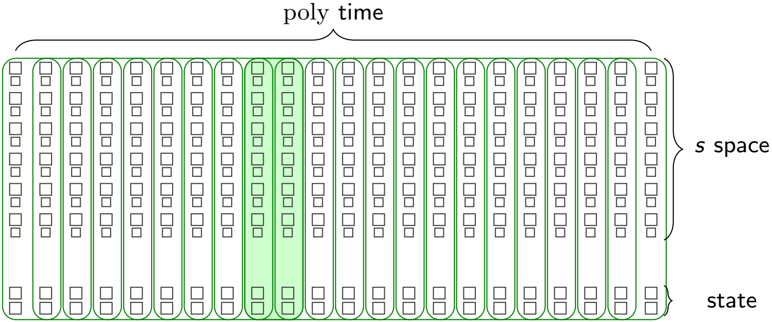

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  iff Directed Reachability can be solved in these bounds.
- In a restricted model we know $2^{o(\log^2 n)}$ time $n^{o(1)}$ space unconditional lower bounds.

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Our results

- tree-depth. . .
- Conjecture:
  "LONGEST COMMON SUBSEQ on input: alphabet, k strings cannot be solved in $n^{f(k)}$ time and $f(k)\mathrm{poly}$ space."

  Related to the *space-efficient* question for `pw`,
  for determinization.
  $\rightarrow$ completeness by [Elberfeld, Stockhusen, Tantau '14]

Tree-depth

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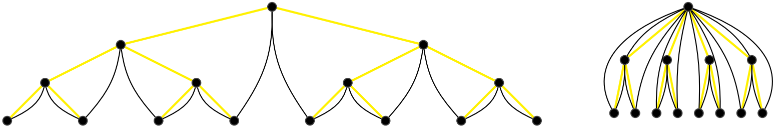

- Definition:
  - Intuitively: # of forget nodes on any root-leaf path of a nice tree decomposition.
  - min depth of a tree whose ancestor-closure contains G.
  - cops and robbers game, but a cop, once placed, cannot move.
  - Many others...
- Motivation:
  - Important in sparsity theory.
  - Some trichotomies for CSP complexity involve $\mathtt{tw}, \mathtt{pw}, \mathtt{td}$.
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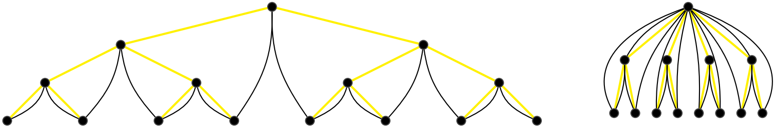

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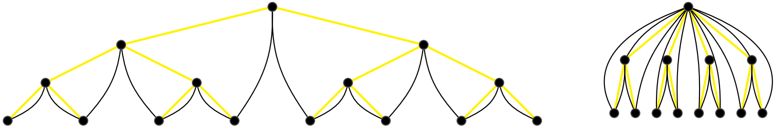

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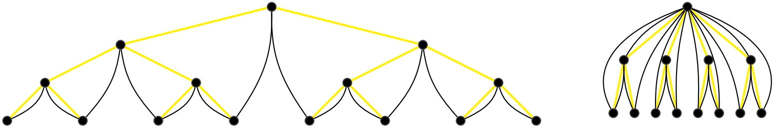

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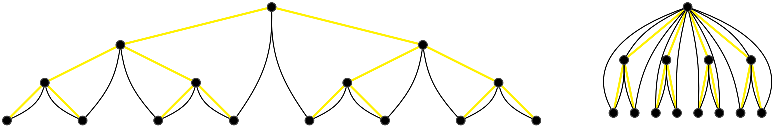

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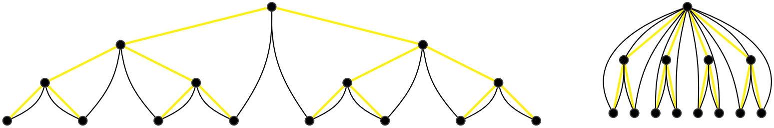

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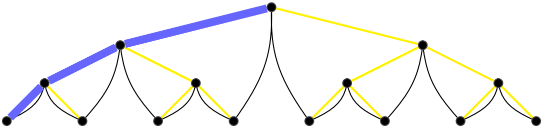

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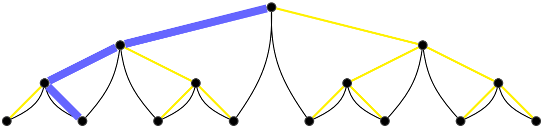

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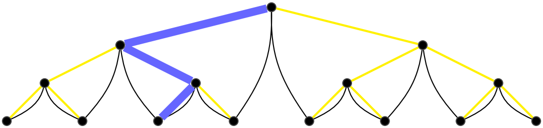

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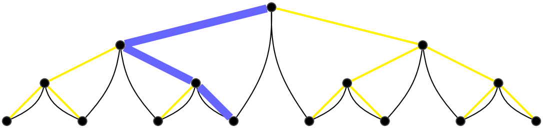

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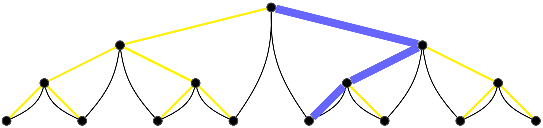

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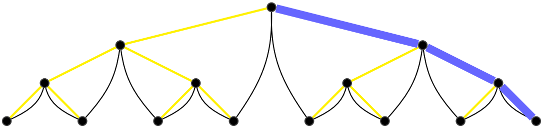
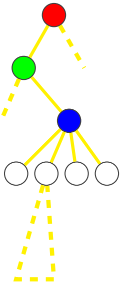


3COL on tree-depth

3COL can be solved in $\mathcal{O}(\mathtt{td} + \log)$ space (so $c^{\mathtt{td}}\mathrm{poly}$ time).



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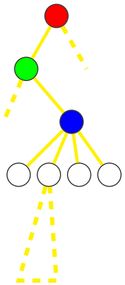

DOMSET can be solved in $3^{\mathtt{td}}\mathrm{poly}$ time and poly space.
→ Möbius transform [Fürer, Yu '14], [Lokshtanov, Nederlof '10]

Completeness for td

pw-SAT[s] is complete for **N[** $\underset{time}{\mathrm{poly}}, \underset{space}{s}$ **]**

tw-SAT[s] is complete for **NAuxPDA[** $\underset{time}{\mathrm{poly}}, \underset{space}{s}$ **]**

Let $s(n)$ be a nice polylogarithmic function $\geq \log^2 n$.

Thm

td-3Col[s] is complete for **NAuxSA[** $\underset{time}{\mathrm{poly}}, \underset{space}{\log}, \underset{height}{s}$ **]**

– nondet machines with poly time, log space and a fully readable **stack of height** s.

Hierarchies

$$\mathbf{NAuxSA}[\underset{time}{\text{poly}}, \underset{space}{\log}, \underset{height}{s}] = [\text{td-3}\textsc{Coloring}[s]]^L$$

$$\cap\!|$$

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Hierarchies – determinization

$$\textbf{NAuxSA}[\underset{time}{\mathrm{poly}}, \underset{space}{\log}, \underset{height}{s}] = [\text{td-3}\textsc{Coloring}[s]]^L \subseteq \textbf{D}[\underset{space}{s}]$$

$$\rotatebox{90}{$\subseteq$}$$

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$$\textbf{NAuxPDA}[\underset{time}{\mathrm{poly}}, \underset{space}{s}] = [\text{tw-3}\textsc{Coloring}[s]]^L \subseteq \textbf{D}[\underset{time}{2^{\mathcal{O}(s)}}]$$

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Corollaries

- $\mathbf{NAuxSA}[\underset{time}{\text{poly}}, \underset{space}{\log}, \underset{height}{s}] \subseteq \mathbf{D}[\underset{space}{s}]$
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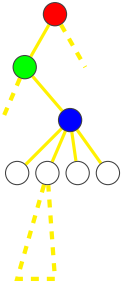
Containement

Solving td-3Col[s] in the **NAuxSA[**$\underbrace{\mathrm{poly}}_{\textit{time}}$, $\underbrace{\log}_{\textit{space}}$, $\underbrace{s}_{\textit{height}}$ **]** model:

- Enter the root
- When entering v:
  - guess its color, push it onto the stack,
  - compare with all ancestors,
  - recurse into each subtree.
- When leaving v: pop its color from the stack.

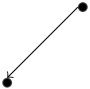


Proof of hardness

Push-pop tree

→ [Akatov, Gottlob '10]

push pop push push push pop push pop pop pop



Push-pop tree

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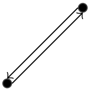

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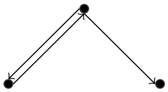

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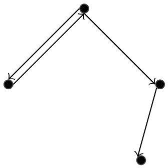

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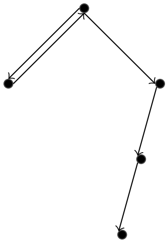

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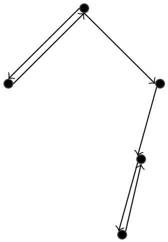

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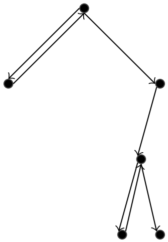

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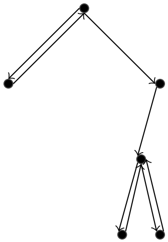

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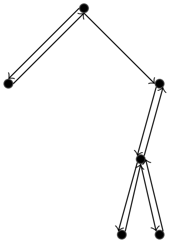

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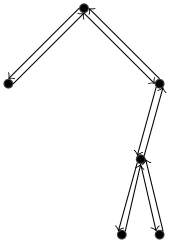

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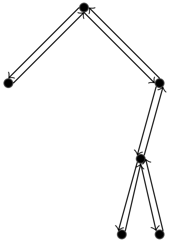


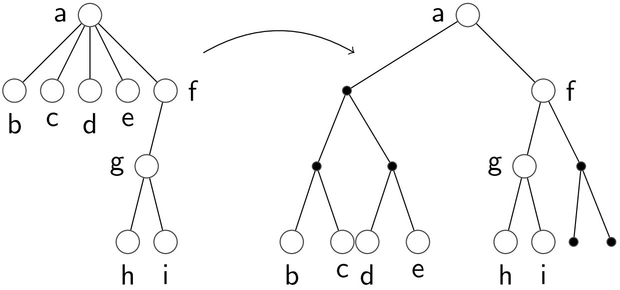
The arrows in order give the *Euler walk* of the tree.

Regularizing the tree

Lemma [Akatov, Gottlob '10], [Elberfeld, Jakoby, Tantau '10]

Every tree T of depth $\leq \lg|T|$ has an *embedding* into a full binary tree of depth $4\lg|T|$.

embedding = injection preserving ancestor relation and Euler walk.



Regularizing the machine

Consider a **NAuxSA[**$\underbrace{\mathrm{poly}}_{time}, \underbrace{\log}_{space}, \underbrace{s}_{height}$**]** machine. W.l.o.g. assume:

- It is a **NAuxSA[**$\underbrace{\mathrm{poly}}_{time}, \underbrace{s/\log}_{space}, \underbrace{s}_{height}$**]** machine,
- It pushes and pop blocks of $\lceil s/\lg \rceil$ symbols at a time, (by keeping the topmost symbols on a buffer on the tape)
- The push-pop tree (atomic 'block' pushes) has depth $\leq \lg n$.
- The push-pop tree is the full binary tree of depth $4 \lg n$.

Regularizing the machine

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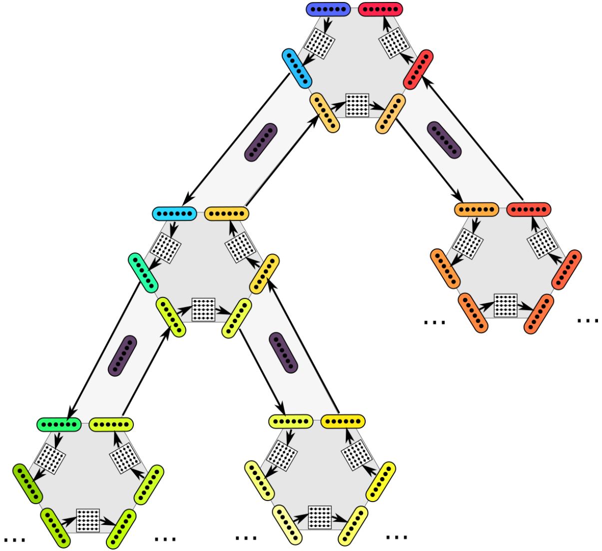
Wrapping computation around the push-pop tree



Conclusions

$$\mathbf{NAuxSA[}\underset{time}{\mathrm{poly}}, \underset{space}{\log}, \underset{height}{s}\mathbf{]} = [\text{td-3}\textsc{Coloring}[s]]^L \subseteq \mathbf{D[}\underset{space}{s}\mathbf{]}$$

$$\rotatebox{90}{$\subseteq$}$$

$$\mathbf{N[}\underset{time}{\mathrm{poly}}, \underset{space}{s}\mathbf{]} = [\text{pw-3}\textsc{Coloring}[s]]^L$$

$$\rotatebox{90}{$\subseteq$}$$

$$\mathbf{NAuxPDA[}\underset{time}{\mathrm{poly}}, \underset{space}{s}\mathbf{]} = [\text{tw-3}\textsc{Coloring}[s]]^L \subseteq \mathbf{D[}\underset{time}{2^{\mathcal{O}(s)}}\mathbf{]}$$

$$\rotatebox{90}{$\subseteq$}$$

$$\mathbf{NAuxSA[}\underset{time}{\mathrm{poly}}, \underset{space}{\log}, \underset{height}{s \cdot \log}\mathbf{]} = [\text{td-3}\textsc{Coloring}[s \cdot \log]]^L \subseteq \mathbf{D[}\underset{space}{s \cdot \log}\mathbf{]}$$

Conclusions

$$\mathbf{NAuxSA}[\underset{time}{\text{poly}}, \underset{space}{\log}, \underset{height}{s}] = [\text{td-3}\textsc{Coloring}[s]]^L \subseteq \mathbf{D}[\underset{space}{s}]$$

$$\cap\!|$$

$$\mathbf{N}[\underset{time}{\text{poly}}, \underset{space}{s}] = [\text{pw-3}\textsc{Coloring}[s]]^L$$

$$\cap\!|$$

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$$\cap\!|$$

$$\mathbf{NAuxSA}[\underset{time}{\text{poly}}, \underset{space}{\log}, \underset{height}{s \cdot \log}] = [\text{td-3}\textsc{Coloring}[s \cdot \log]]^L \subseteq \mathbf{D}[\underset{space}{s \cdot \log}]$$

Thank you!