The structure of graphs excluding Gem and $\hat{K}_4$ as induced minors

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[4]LIP, ENS de Lyon, France

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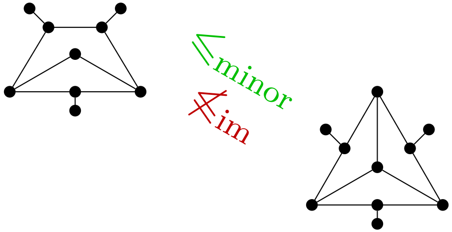

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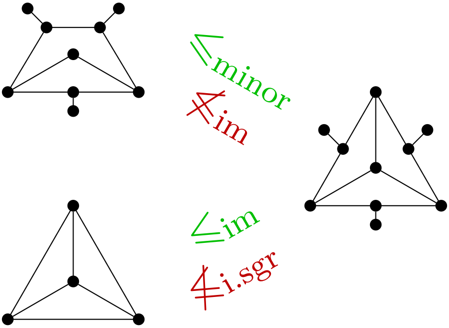

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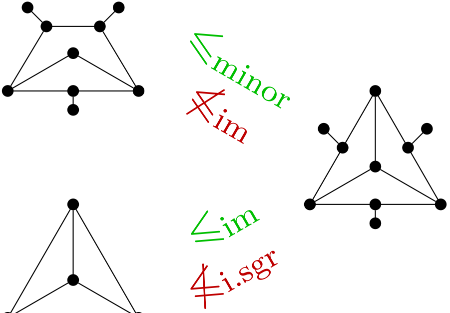
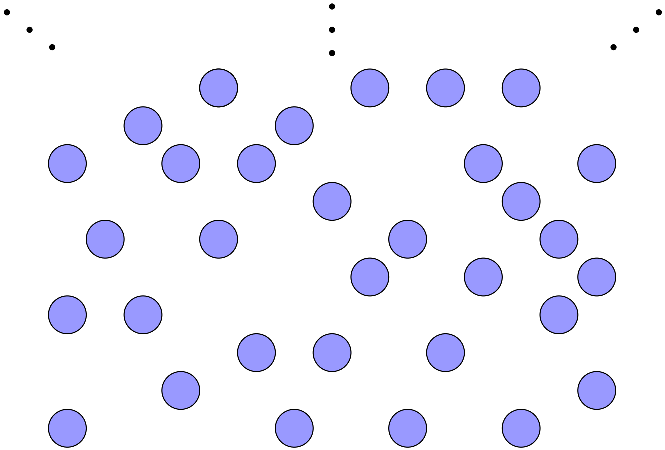


between minors and induced subgraphs

Excluding a graph



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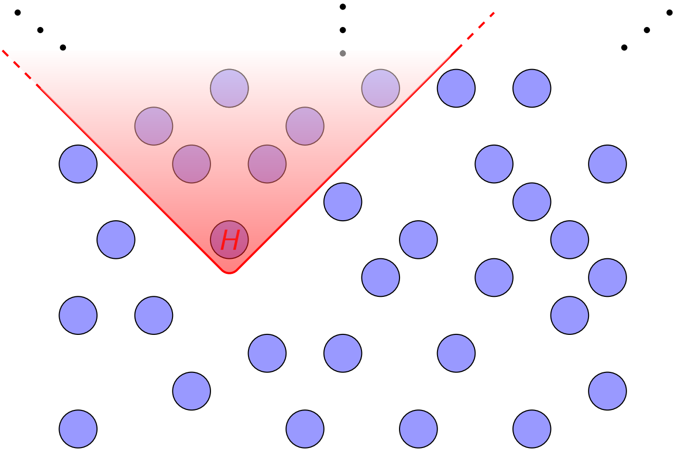

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Theorem (Excluded minor theorem, Robertson and Seymour)

Every H-minor-free graph can be obtained by clique sums of order at most c_H from graphs that can be c_H-nearly embedded in some surface, in which H cannot be embedded.

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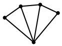
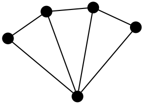
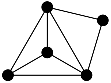
Our contribution: structure of graphs excluding  /  as induced minors.

The main characters

Gem = 

$\hat{K}_4$ = 

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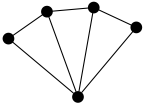
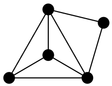
$\mathrm{Gem} =$ 

$\hat{K}_4 =$ 

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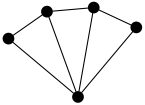
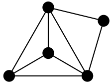
$\mathrm{Gem} =$ 

$\hat{K}_4 =$ 

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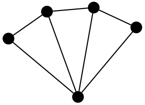
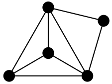
Gem = 

$\hat{K}_4$ = 

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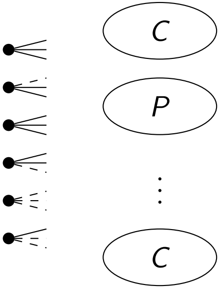
Why them?

- excluding each of them yields a well-quasi-order (wrt. $\leqslant_{\mathrm{im}}$);
- we use the decomposition theorems to prove the wqo result.

Decomposition of Gem-induced minor-free graphs

Theorem (Błasiok, Kamiński, R., Trunck, 2015+)

If $\mathrm{Gem} \not\leqslant_{\mathrm{im}} G$ and G is 2-c, removing some $\leqslant 6$ vertices gives a disjoint union of cographs and (straight) paths.



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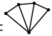
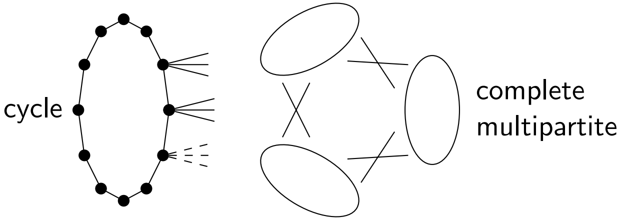
(Reminder: $\mathrm{Gem} =$ )

Theorem (Błasiok, Kamiński, R., Trunck, 2015+)

If $\hat{K}_4 \not\leqslant_{\mathrm{im}} G$ and G is 2-c, then

- *either $K_4 \not\leqslant_{\mathrm{im}} G$;*
- *or G is a subdivision of a graph on at most 9 vertices;*
- *or $V(G) = M \cup C$ where $G[M]$ is complete multipartite, $G[C]$ is a cycle and adjacencies are "binary".*



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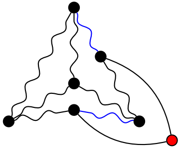

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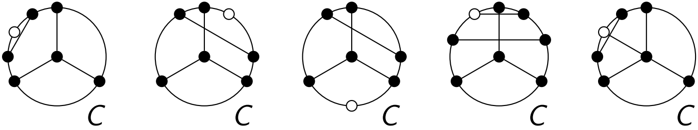

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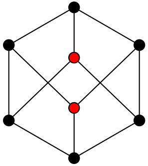

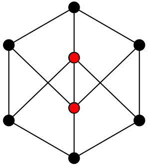

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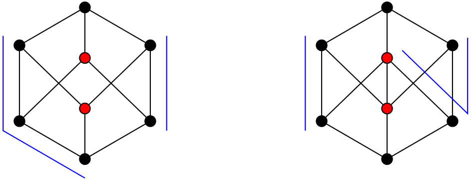

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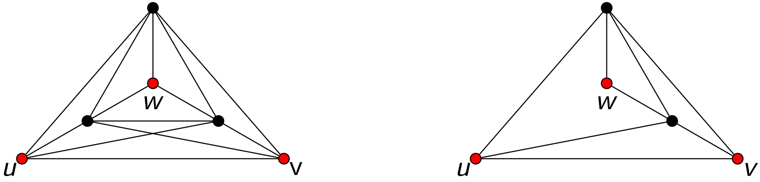

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- $\rightarrow$ cycle-multipartite decomposition

Gem-induced minor-free graphs (reminder)

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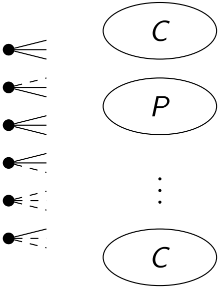

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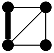
The rooted diamond: 

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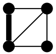
The rooted diamond: 

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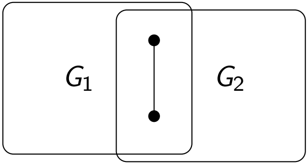

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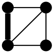
The rooted diamond: 

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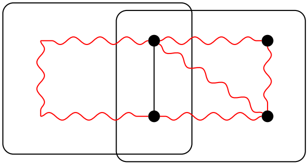

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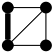
The rooted diamond: 

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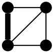
The rooted diamond: 

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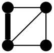
The rooted diamond: 

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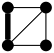
The rooted diamond: 

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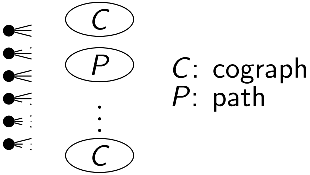

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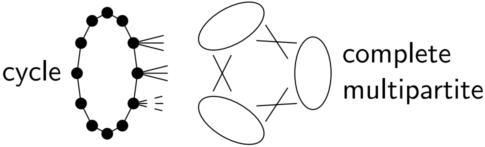

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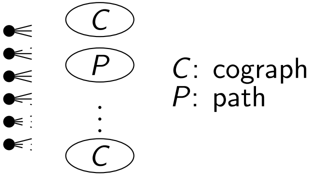

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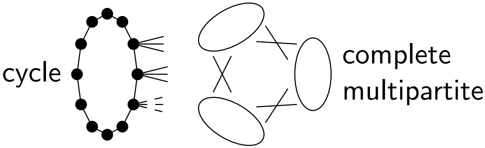

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Note: structural results on these classes of graphs have also been obtained recently by Belmonte et al..

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