1-Sperner hypergraphs and new characterizations of threshold graphs

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Joint work with

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Background and motivation

Hypergraphs

Hypergraph: a pair $\mathcal{H} = (V, \mathcal{E})$ where

- V is a finite set of **vertices**
- $\mathcal{E}$ is a set of subsets of V, called **hyperedges**

Example:

- $V = \{1, 2, 3, 4\}$
- $\mathcal{E} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$

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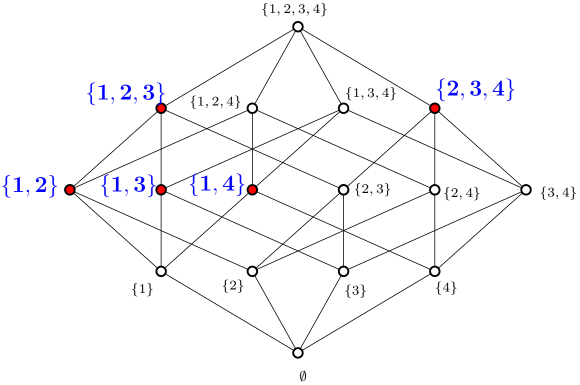

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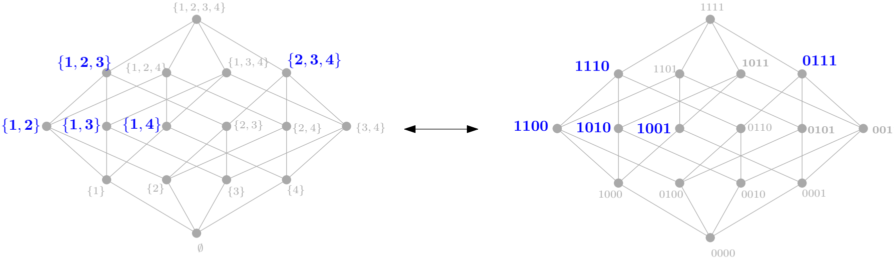


set $S \subseteq V$ $\longleftrightarrow$ $\chi^S \in \{0,1\}^V$

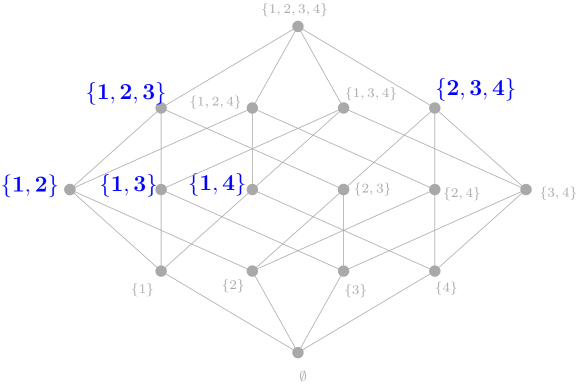
characteristic vector of S

Hypergraphs

An **independent set** in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a set $X \subseteq V$ containing no hyperedge of $\mathcal{H}$.

A set $X \subseteq V$ is **dependent** if it is not independent.

Example:



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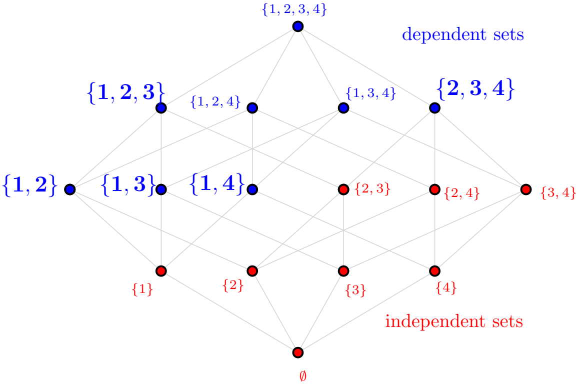

Threshold hypergraphs

A hypergraph $\mathcal{H}$ is **threshold** if $\exists\ w : V \to \mathbb{Z}_{\geq 0}, \exists\ t \in \mathbb{Z}_{\geq 0}$ such that

for all $X \subseteq V$:

$$X \text{ is a dependent set of } \mathcal{H} \quad \Leftrightarrow \quad w(X) \geq t$$

$w(X) := \sum_{x \in X} w(x)$.

- **Geometric interpretation:** *there is a hyperplane separating the characteristic vectors of independent sets from the characteristic vectors of dependent sets.*

Threshold hypergraphs - example

$\exists\ w : V \to \mathbb{Z}_{\geq 0}, \exists\ t \in \mathbb{Z}_{\geq 0}$ such that
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Example:

$\{1,2,3\}$ $\{2,3,4\}$

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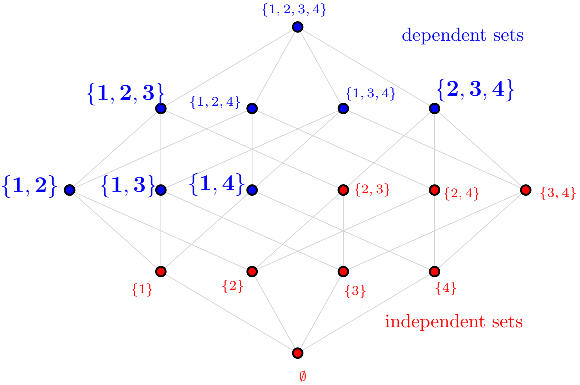

Threshold hypergraphs - example

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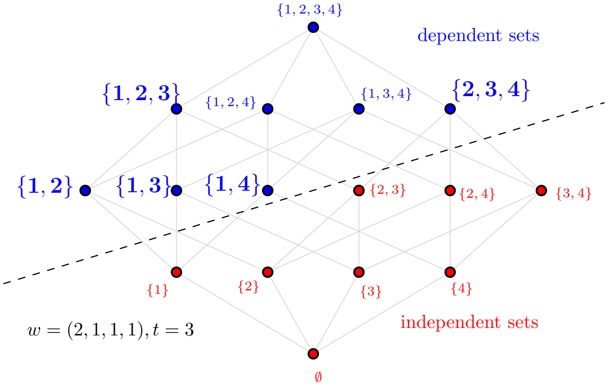


Threshold hypergraphs

Some historical remarks:

- Threshold hypergraphs were defined in the uniform case by Golumbic in 1980 and studied further by Reiterman, Rödl, Šiňajová, Tůma in 1985.
- In their full generality, the concept of threshold hypergraphs is equivalent to that of **threshold monotone Boolean functions** studied, e.g., by Muroga in 1971.

Threshold hypergraphs

- A polynomial time recognition algorithm for threshold monotone Boolean functions represented by their complete DNF was given by Peled and Simeone in 1985.
- The algorithm is based on linear programming and implies the existence of a **polynomial time recognition algorithm for threshold hypergraphs**.

Sperner hypergraphs

A hypergraph is said to be **Sperner** (or: a **clutter**) if no hyperedge contains another one, that is,

if $e, f \in \mathcal{E}$ and $e \subseteq f$ implies $e = f$.

Example:

- $V = \{1, 2, 3, 4\}$
- $\mathcal{E} =$

$\{1,2,3\}$ $\{2,3,4\}$

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$

not Sperner since $\{1, 2\} \subset \{1, 2, 3\}$

Dually Sperner hypergraphs

Sperner hypergraphs can be equivalently defined as the hypergraphs such that

$$e \neq f \quad \Rightarrow \quad \min\{|e \setminus f|, |f \setminus e|\} \geq 1\,.$$

This point of view motivated Chiarelli and Milanič to define in 2014 a hypergraph $\mathcal{H}$ to be **dually Sperner** if

$$e \neq f \quad \Rightarrow \quad \min\{|e \setminus f|, |f \setminus e|\} \leq 1\,.$$

Dually Sperner hypergraphs

$$e \neq f \quad \Rightarrow \quad \min\{|e \setminus f|, |f \setminus e|\} \leq 1\,.$$

Example:

The hypergraph from the previous example is dually Sperner:

- $\mathcal{E} =$

$\{1,2,3\}$ $\{2,3,4\}$

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$

The following hypergraph is not dually Sperner:

- $V = \{1,2,3,4\}$
- $\mathcal{E} = \{\{1,2\},\{3,4\}\}$

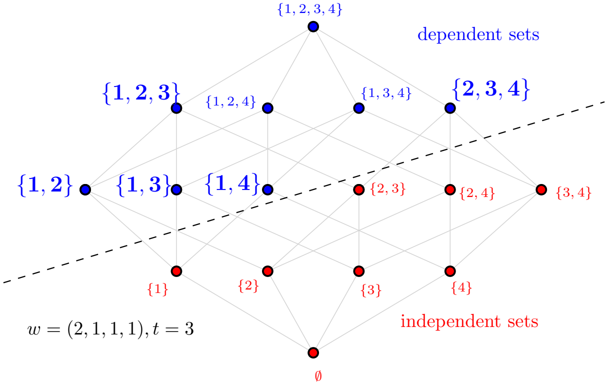
Dually Sperner hypergraphs

Theorem (Chiarelli-M. 2014)
Every dually Sperner hypergraph is threshold.

Chiarelli and Milanič applied dually Sperner hypergraphs to characterize two classes of graphs related to separation of total, resp. connected dominating sets.

Threshold hypergraphs

A hypergraph $\mathcal{H}$ is **threshold** if
$\exists\, w : V \to \mathbb{Z}_{\geq 0}, \exists\, t \in \mathbb{Z}_{\geq 0}$ such that
for all $X \subseteq V$:
X is an dependent set of $\mathcal{H} \quad \Leftrightarrow \quad w(X) \geq t$.



Threshold hypergraphs

It follows from the definition of threshold hypergraphs that **only minimal hyperedges matter** for the thresholdness property of a given hypergraph.

Example:

$\{1,2,3\}$ $\{2,3,4\}$

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$

is threshold if and only if

$\{2,3,4\}$

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$

is threshold.

1-Sperner hypergraphs

Since dually Sperner hypergraphs are threshold, we focus on the family of hypergraphs that are **both Sperner and dually Sperner**.

We call such hypergraphs 1**-Sperner.**

A hypergraph $\mathcal{H}$ is 1-Sperner if and only if

$$e \neq f \quad \Rightarrow \quad \min\{|e \setminus f|, |f \setminus e|\} = 1\,.$$

1-Sperner hypergraphs

$$e \neq f \quad \Rightarrow \quad \min\{|e \setminus f|, |f \setminus e|\} = 1\,.$$

Example:
The hypergraph from the previous example is not 1-Sperner, since it is not Sperner:

- $\mathcal{E} =$

$\{1,2,3\}$ $\{2,3,4\}$

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$

Deleting the hyperedge $\{1, 2, 3\}$ results in a 1-Sperner hypergraph:

- $V = \{1, 2, 3, 4\}$
- $\mathcal{E} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$

Our results

Our results

1. A composition result for 1-Sperner hypergraphs.
2. Its consequences.
3. New characterizations of threshold graphs.

An operation preserving 1-Spernerness

Gluing of hypergraphs

$\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ – vertex-disjoint hypergraphs

z – a new vertex

The **gluing of** $\mathcal{H}_1$ **and** $\mathcal{H}_2$ is the hypergraph

$$\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$$

such that

$$V(\mathcal{H}) = V_1 \cup V_2 \cup \{z\}$$

and

$$E(\mathcal{H}) = \{\{z\} \cup e \mid e \in \mathcal{E}_1\} \cup \{V_1 \cup e \mid e \in \mathcal{E}_2\} .$$

Incidence matrices

The operation of gluing can be visualized easily in terms of **incidence matrices**.

Every hypergraph $\mathcal{H} = (V, \mathcal{E})$ with
$V = \{v_1, \ldots, v_n\}$ and $\mathcal{E} = \{e_1, \ldots, e_m\}$

can be represented with its
incidence matrix $A^{\mathcal{H}} \in \{0, 1\}^{m \times n}$:
rows are indexed by hyperedges of $\mathcal{H}$,
columns are indexed by vertices of $\mathcal{H}$,

and

$$A^{\mathcal{H}}_{i,j} = \begin{cases} 1, & \text{if } v_j \in e_i; \\ 0, & \text{otherwise.} \end{cases}$$

Gluing of hypergraphs

If $\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2$ then

$$A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \begin{pmatrix} \mathbf{1}^{m_1,1} & A^{\mathcal{H}_1} & \mathbf{0}^{m_1,n_2} \\ \mathbf{0}^{m_2,1} & \mathbf{1}^{m_2,n_1} & A^{\mathcal{H}_2} \end{pmatrix}.$$

Example:

$$A^{\mathcal{H}_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{\mathcal{H}_2} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\longrightarrow$$

$$A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \left(\begin{array}{c|cc|ccc} z & & & & & \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$$

Gluing of hypergraphs

Proposition

For every pair $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ of vertex-disjoint 1-Sperner hypergraphs,
their gluing $\mathcal{H}_1 \odot \mathcal{H}_2$ is a 1-Sperner hypergraph,
unless $\mathcal{E}_1 = \{V_1\}$ and $\mathcal{E}_2 = \{\emptyset\}$
(in which case $\mathcal{H}_1 \odot \mathcal{H}_2$ is not Sperner).

$$A^{\mathcal{H}_1} = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$A^{\mathcal{H}_2} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow$$

$$A^{\mathcal{H}_1 \odot \mathcal{H}_2} = \left(\begin{array}{c|cc|ccc} \overset{z}{1} & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

A composition result for 1-Sperner hypergraphs

We show that every nontrivial 1-Sperner hypergraph can be generated this way.

We say that a gluing of two vertex-disjoint hypergraphs $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$ is **safe** unless $\mathcal{E}_1 = \{V_1\}$ and $\mathcal{E}_2 = \{\emptyset\}$.

Theorem

A hypergraph $\mathcal{H}$ is 1-Sperner if and only if

it either has no vertices (that is, $\mathcal{H} \in \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\})\}$)

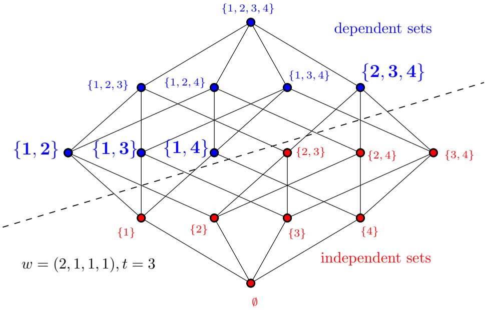
or it is a safe gluing of two smaller 1-Sperner hypergraphs.

Consequences of the structural result

Consequences

Using the composition result for 1-Sperner hypergraph, we obtain the following:

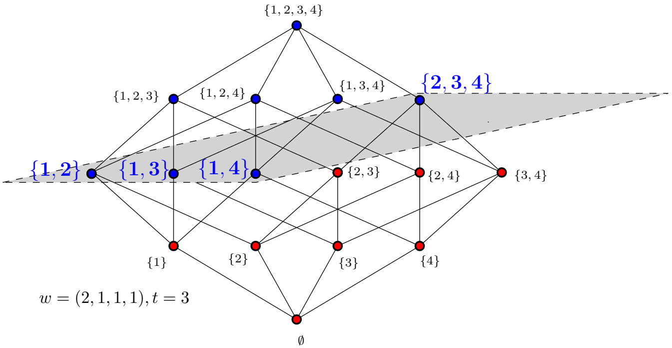
1. An alternative proof of the fact that every 1-Sperner hypergraph is threshold.



- Unlike the previous proof establishing thresholdness of dually Sperner hypergraphs (due to Chiarelli-M.), this proof is constructive and builds a separating hyperplane of a given 1-Sperner hypergraph.

Consequences

2. A proof of the fact that every 1-Sperner hypergraph is equilizable.



- Equilizable hypergraphs form a generalization of **equistable graphs** (introduced in 1980 by Payan and studied afterwards in more than 10 papers).

Consequences

3. An upper bound on the size of 1-Sperner hypergraphs:

Proposition

For every 1-Sperner hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $\mathcal{E} \neq \{\emptyset\}$, we have $|\mathcal{E}| \leq |V|$.

- Proof idea: the characteristic vectors of the hyperedges are linearly independent in $\mathbb{R}^V$.

Can we prove a similar **lower bound?**

Consequences

- **universal** vertex: a vertex contained in all hyperedges
- **isolated** vertex: a vertex contained in no hyperedges
- two vertices u, v are **twins** if they are contained in exactly the same hyperedges

Adding universal vertices, isolated vertices, or twin vertices preserves the 1-Sperner property, while

- keeping the number of hyperedges unchanged and
- increasing the number of vertices.

Consequently, there is no lower bound on the number of hyperedges of a 1-Sperner hypergraph in terms of the number of vertices.

However ...

Consequences

4. A lower bound on the size of 1-Sperner hypergraphs without universal, isolated, and twin vertices:

Proposition

For every 1*-Sperner hypergraph* $\mathcal{H} = (V, \mathcal{E})$ *with* $|V| \geq 2$ *and without universal, isolated, and twin vertices, we have*

$$|\mathcal{E}| \geq \left\lceil \frac{|V| + 2}{2} \right\rceil .$$

This bound is sharp.

New characterizations of threshold graphs

Threshold graphs

A **threshold graph** is a threshold hypergraph in which all hyperedges are of size 2.

- Threshold graphs were introduced by Chvátal and Hammer in the 1970s and were studied in numerous papers (and in a monograph by Mahadev and Peled from 1995).
- Threshold graphs have many different characterizations.

Threshold graphs

Theorem (Chvátal and Hammer, 1977)
A graph G is threshold if and only if it is $\{P_4, C_4, 2K_2\}$*-free.*

Theorem (Chvátal and Hammer, 1977)
A graph G is threshold if and only if $V(G) = K \cup I$

where K *is a clique,* I *is an independent set,* $K \cap I = \emptyset$*, and*

there exists an ordering $v_1, \ldots, v_k$ *of* I *such that* $N(v_i) \subseteq N(v_j)$ *for all* $1 \leq i < j \leq k$*.*

Clique hypergraphs of graphs

Given a graph G, the **clique hypergraph of** G is the hypergraph $\mathcal{C}(G)$ with vertex set $V(G)$ in which the hyperedges are exactly the maximal cliques of G.

Theorem (Berge, 1989)

The clique hypergraphs of graphs are exactly those Sperner hypergraphs $\mathcal{H}$ *that are also* **normal** *(or:* **conformal***),*
that is,
for every set $X \subseteq V(\mathcal{H})$ *such that every pair of elements in* X *is contained in a hyperedge,*
there exists a hyperedge containing X*.*

A necessary condition for thresholdness

A hypergraph is k**-summable** if it has
k (not necessarily distinct) independent sets $A_1, \ldots, A_k$
and k (not necessarily distinct) dependent sets $B_1, \ldots, B_k$
such that

$$\sum_{i=1}^{k} \chi^{A_i} = \sum_{i=1}^{k} \chi^{B_i} .$$

If a graph is k-summable for some $k \geq 2$, then it cannot be threshold.

A hypergraph is k**-asummable** if it is not k-summable.

A necessary condition for thresholdness

Theorem

A hypergraph is threshold if and only if it is k-asummable for all k.

- A restatement of the analogous characterization of threshold Boolean functions proved in 1961 independently by Chow and Elgot.

Corollary

Every threshold hypergraph is 2-asummable.

1-Sperner, threshold, 2-asummable

In general:

$$1\text{-Sperner} \quad \Rightarrow \quad \text{threshold} \quad \Rightarrow \quad 2\text{-asummable}$$

and none of the implications can be reversed.

In the class of **conformal** Sperner hypergraphs, all these three notions coincide.

Moreover, they exactly characterize threshold graphs.

New characterizations of threshold graphs

Theorem
For every graph G, the following statements are equivalent:

(1) *G is threshold.*

(2) *The clique hypergraph $\mathcal{C}(G)$ is 1-Sperner.*

(3) *The clique hypergraph $\mathcal{C}(G)$ is threshold.*

(4) *The clique hypergraph $\mathcal{C}(G)$ is 2-asummable.*

clique hypergraph $\mathcal{C}(G)$
$\rightsquigarrow$ **independent set hypergraph** $\mathcal{I}(G)$
in (2), (3), (4) also ok

(since the class of threshold graphs is closed under taking complements)

Summary

- We introduced a new class of hypergraphs, the class of 1**-Sperner hypergraphs**:
  $e \neq f \quad \Rightarrow \quad \min\{|e \setminus f|, |f \setminus e|\} = 1\,.$
- We proved a **structural theorem** for 1-Sperner hypergraphs and examined several of its **consequences**, including bounds on the size of 1-Sperner hypergraphs and a new, constructive proof of the fact that every 1-Sperner hypergraph is threshold.
- Within the class of **normal Sperner hypergraphs**:
  1-Sperner $\Leftrightarrow$ threshold $\Leftrightarrow$ 2-asummable
- New characterizations of the class of threshold graphs.

T$\mathcal{H}$ank you!

M$\mathcal{E}$rci!