# Polynomial Fixed-Parameter Algorithms: A Case Study for Longest Path on Interval Graphs

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# Fixed-parameter algorithms

For many combinatorial problems the best known (exact) algorithms are too slow:

- exponential running times (for NP-hard problems)
- polynomials of high degree, e.g. $O(n^3)$, $O(n^4)$, ... (for problems in P)

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The successful "FPT approach" for exact computation:

- identify an appropriate parameter $k$ that "causes" large running times
- design algorithms that separate the dependency of the running time from the input size $n$ and the parameter $k$

More formally:

- a fixed-parameter algorithm solves a problem with input size $n$ and parameter $k$ in $f(k) \cdot n^{O(1)}$ time

$\Rightarrow$ whenever $k$ is small, the algorithm is efficient for every input size $n$

# Fixed-parameter algorithms

- Fixed-Parameter Tractability (FPT) is a flourishing field, see e.g.
  [Downey, Fellows, *Parameterized Complexity*, 1999]
  [Flum, Grohe, *Parameterized Complexity Theory*, 2006]
  [Niedermeier, *Invitation to Fixed-Parameter Algorithms*, 2006]
  [Downey, Fellows, *Fundamentals of Parameterized Complexity*, 2013]
  [Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk$^2$, Saurabh, *Parameterized Algorithms*, 2015]
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- There is a growing awareness about the polynomial factors $n^{O(1)}$ (which were usually neglected), e.g.:
  - computing the treewidth: [Bodlaender, *SIAM J. on Computing*, 1996]
  - computing the crossing number: [Kawarabayashi, Reed, *STOC*, 2007]
  - problems from industrial applications: [van Bevern, *PhD Thesis*, 2014]
  - these works emphasize "linear time" in the title, instead of "FPT"

# “FPT inside P”

- Although polynomially solvable problems are theoretically tractable:
  - often the best known algorithms are not efficient in practice, e.g.
  - Linear Programming on arbitrary instances (interior point algorithms)
  - Matrix Multiplication (currently in $O(n^{2.373})$ time)
  - Maximum Matching (in $O(m\sqrt{n})$ time worst-case)

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- Reducing the worst-case complexity:
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- Reducing the worst-case complexity:
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- Towards reducing polynomial factors $n^{O(1)}$:
  - the "FPT approach" can help refining the complexity of problems in P
- Appropriate parameterizations of a problem within P:
  - can reveal what makes it "far from being solvable in linear time"
  - in the same spirit as classical FPT algorithms (why is it "far from P")

# "FPT inside P"

Formally, given a problem $\Pi$ with instance size $n$:

- for which there exists an $O(n^c)$-time algorithm

we aim at detecting an appropriate parameter $k$ such that:

- there exists an $f(k) \cdot n^{c'}$-time algorithm where
  1. $c' < c$ and
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## Definition (refinement of FPT)

For every polynomially bounded function $p(n)$, the class $\text{FPT}(p(n))$ contains the problems solvable in $f(k) \cdot p(n)$ time, where $f(k)$ is an arbitrary (possibly exponential) function of $k$.

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For a problem within P:

- it is possible that $f(k)$ can become polynomial on $k$
- in wide contrast to FPT algorithms for NP-hard problems!

# “FPT inside P”

Motivated by this:

## Definition (refinement of P)

For every polynomially bounded function $p(n)$, the class P-FPT$(p(n))$ *(Polynomial Fixed-Parameter Tractable)* contains the problems solvable in $O(k^t \cdot p(n))$ time for some constant $t \geq 1$, i.e. $f(k) = k^t$.

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For the case where $p(n) = n$, the class P-FPT$(n)$ is called PL-FPT *(Polynomial-Linear Fixed-Parameter Tractable)*.

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This "FPT inside P" theme:

- interesting research direction
- too little explored so far
- few known results, scattered around in the literature

# "FPT inside P"

We propose three desirable algorithmic properties:

1. the running time should depend polynomially on the parameter $k$
   $\Rightarrow$ the problem is in P-FPT$(p(n))$, for some polynomial $p(n)$
2. when $k$ is constant, the running time should be as close to linear as possible
   $\Rightarrow$ the problem is in PL-FPT, or at least in P-FPT$(p(n))$ where $p(n) \approx n$
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The "FPT inside P" framework should be systematically studied:

- exploiting the rich toolbox of parameterized algorithm design
  - e.g. data reductions, kernelization, ...
- having these three properties as a "compass"

# Related work

## Shortest path problems

- Some polynomial algorithms can be "tuned" with respect to specific parameters:
  - classic Dijkstra's algorithm for shortest paths: $O(m + n \log n)$ time
  - can be adapted to: $O(m + n \log k)$ time, where $k$ is the number of distinct edge weights
    [Orlin, Madduri, Subramani, Williamson, *J. of Discr. Alg.*, 2010]
    [Koutis, Miller, Peng, *FOCS*, 2011]

- In order to prove the efficiency of known heuristics for road networks:
  - the parameter highway dimension has been introduced
    [Abraham, Fiat, Goldberg, Werneck, *SODA*, 2010]
  - Dijkstra's algorithm is too slow in practice

Conclusion: Adopting a parameterized view may be of significant practical interest, even for quasi-linear algorithms

# Related work

## Maximum flow problems

- For graphs made planar by deleting $k$ crossing edges:
  - maximum flow in $O(k^3 \cdot n \log n)$ time [Hochstein, Weihe, *SODA*, 2007]
  - an embedding and the $k$ crossing edges are given in the input

  $\Rightarrow$ this violates Property 3 (no known good approximation of $k$)

- For graphs with bounded genus $g$ and sum of capacities $C$:
  - maximum flow in $O(g^8 \cdot n \log^2 n \; log^2 C)$ time
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  - an embedding and the genus $g$ are given in the input
  - $\Rightarrow$ this violates Property 3 (no known good approximation of $g$)
- Furthermore, when parameterized by the treewidth $k$:
  - multiterminal flow in linear time [Hagerup, Katajainen, Nishimura, Ragde, *J. Comp. & Syst. Sci*, 1998]
  - Wiener index in near-linear time [Cabello, Knauer, *Comp. Geom.*, 2009]
  - both with exponential dependency on $k$
  - $\Rightarrow$ this violates Property 1 (exponential $f(k)$)

- Due to a famous result of Megiddo [Megiddo, *J. of the ACM*, 1984]:
  - Linear Programming in linear time for fixed dimension $d$ (# variables)
  - the multiplicative factor is $f(d) = 2^{O(2^d)}$

$\Rightarrow$ this violates Property 1 (exponential $f(k)$), but is still in P-FPT($n$)

$\Rightarrow$ no guarantee for practically efficient algorithms

  - can be seen as a precursor of "FPT inside P"

# Related work

## Stringology

- String Matching with $k$ Mismatches:
  - "find in a length-$n$ string all occurrences of a length-$m$ pattern with at most $k$ errors"
  - in $O(m^2 + nk^2)$ [Landau, Vishkin, *FOCS*, 1985]
  - in $O(m \log k + nk^2)$ [Landau, Vishkin, *J. Comp. & Syst. Sci*, 1988]
  - in $O(nk)$ [Landau, Vishkin, *J. of Algorithms*, 1989]
  - in $O(n\sqrt{k \log k})$ [Amir, Lewenstein, Porat, *J. of Algorithms*, 2004]
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- All these algorithms are linear in $n$
  - as also in the extreme case $k = 0$ errors
- The parameter $k$ is directly defined by the problem itself (and given with the input)
- Our approach goes beyond that:
  - we try to detect the appropriate parameter that causes a high polynomial time complexity

# Our results

1. A "proof of concept" example: kernelization of `Maximum Matching`
   - parameter $k =$ solution size
   - there exists a "Buss-like" kernel with $O(k^2)$ vertices and edges
   - it can be computed in $O(kn)$ time

$\Rightarrow$ total running time: $O(kn + k^3)$

$\Rightarrow$ `Maximum Matching` is in PL-FPT for parameter $k$

# Kernelization of `Maximum Matching`

## An illustrative example

A kernelization algorithm similar to Buss's for `Vertex Cover`:

- parameter $k$ = solution size

### Reduction Rule 1

*If* $\deg(v) > 2(k-1)$ *for some* $v \in V(G)$ *then return* $(G \setminus \{v\}, k-1)$.

Safeness (idea): if $(G \setminus \{v\}, k-1)$ is a YES-instance, then adding $v$ can always produce a matching of size $\geq k$

- in a matching of size $k-1$ in $G \setminus \{v\}$, there is always "one more edge" in $G$

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### Reduction Rule 2

*If* $\deg(v) = 0$ *for some* $v \in V(G)$ *then return* $(G \setminus \{v\}, k)$.

Safeness: trivial

# Kernelization of Maximum Matching

## An illustrative example

Iteratively apply Reduction Rule 1:

- in total $O(kn)$ time

$\Rightarrow$ $0 \leq \deg(v) \leq 2(k-1)$ for every (remaining) vertex $v$

Iteratively apply Reduction Rule 2:

- again in total $O(kn)$ time

$\Rightarrow$ $1 \leq \deg(v) \leq 2(k-1)$ for every (remaining) vertex $v$

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We can easily prove for the remaining graph $G'$:

**Lemma**

$|V(G')|, |E(G')| \leq (2k-1) \cdot \mathbf{mm}(G')$.

where $\mathbf{mm}(G')$ = size of maximum matching in $G'$

# Kernelization of `Maximum Matching`

## An illustrative example

Putting things together:

- compute the reduced graph $G'$ (by Red. Rules $1 + 2$)
  - in total $O(kn)$ time
- suppose we remove $r$ vertices by Reduction Rule 1
  - if $r \geq k$ then stop and return `YES`
  - else $k' = k - r$
- if $G'$ has more than $(k' - 1)(2k' - 1)$ vertices or edges
  - then stop and return `YES`
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The best known worst-case algorithm:

- in $O(m\sqrt{n}) = O(k^3)$ time [Micali, Vazirani, *FOCS*, 1980]

$\Rightarrow$ total running time: $O(kn + k^3)$ time

# Our results

2. Main technical result: `Longest Path` on `Interval Graphs`
   - `Longest Path` is polynomially solvable in several "small" graph classes:
     - weighted trees, block graphs, ptolemaic graphs, cacti, threshold graphs [Uehara, Uno, *ISAAC*, 2004]

   and only in a few "non-trivial" graph classes:
     - interval graphs, cocomparability graphs, both in $O(n^4)$ time [Ioannidou, Mertzios, Nikolopoulos, *Algorithmica*, 2011] [Mertzios, Corneil, *SIAM J. on Discrete Mathematics*, 2012]

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   - On proper interval graphs:
     - trivially solvable in linear time
     - connected $\Rightarrow$ Hamiltonian

$\Rightarrow$ parameter distance to triviality:
   - $k$ = proper interval (vertex) deletion number
   - $k$ can be 4-approximated in $O(n+m)$ time

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   ⇒ parameter distance to triviality:
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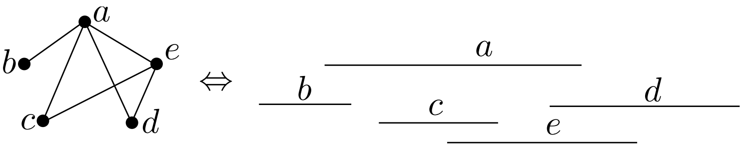
Our Algorithm: compute a longest path in $O(k^9 n)$ time

⇒ Longest Path on Interval Graphs is in PL-FPT for parameter $k$

# Longest Path on Interval Graphs

**Definition**

A graph $G$ is called an interval graph, if $G$ is the intersection graph of a set of intervals on the real line.



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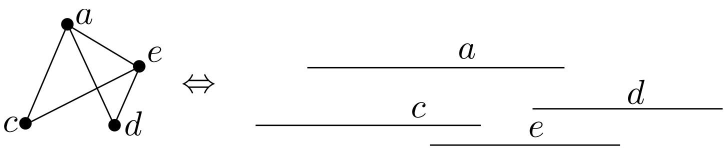
An interval graph $G$ is a proper interval graph, if there exists an interval representation of $G$ where no interval is properly included in another one.



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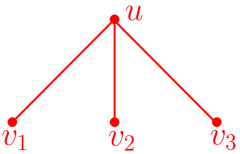
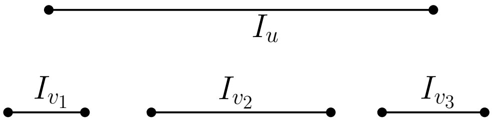
**Theorem (Roberts, 1969)**

*An interval graph $G$ is a proper interval graph $\Longleftrightarrow$ $G$ does not include any claw $K_{1,3}$ as induced subgraph.*

$K_{1,3}$ :



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Computation of a minimum proper interval deletion set $D$:

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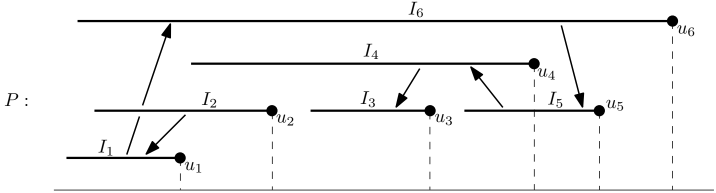
We compute a 4-approximation of $|D|$ in $O(n + m)$ time:

- scan from left to right in the interval representation
- detect a claw $K_{1,3}$
- remove all 4 vertices of the claw
- iterate

- Our proofs are based on the notion of normal paths in interval graphs.
  [Ioannidou, Mertzios, Nikolopoulos, *Algorithmica*, 2011]
  (a.k.a. straight paths: [Damaschke, *Discr. Math*, 1993])

- Main idea:
  - start with the leftmost vertex of the path
  - always continue with the leftmost unvisited neighbor of the current vertex

Example: path $P = (u_2, u_1, u_6, u_5, u_4, u_3)$



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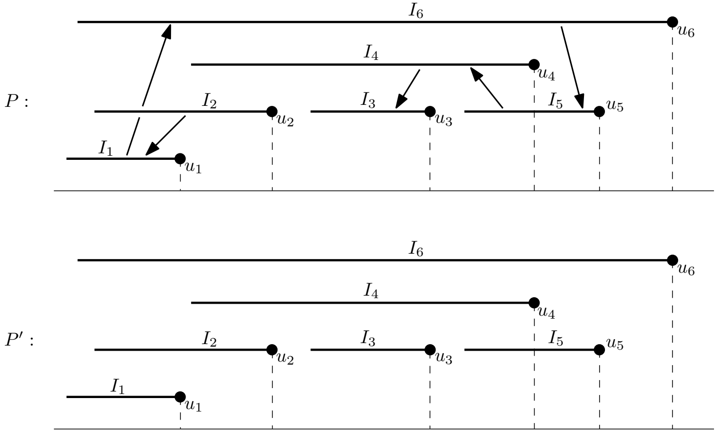


Normal path: $P' = ( \quad , \quad , \quad , \quad , \quad , \quad )$

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## Normal paths in interval graphs

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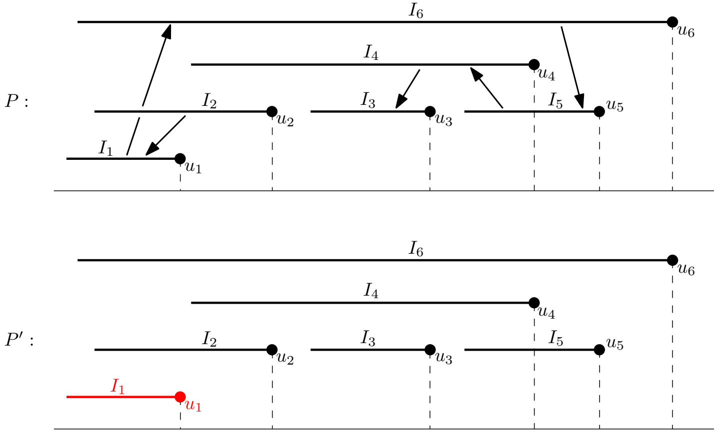


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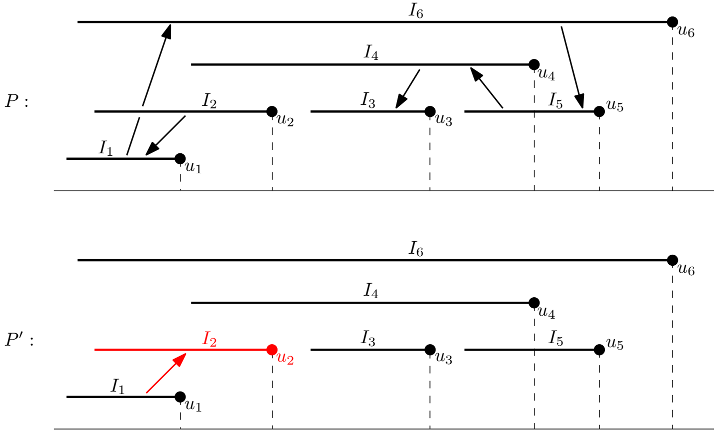


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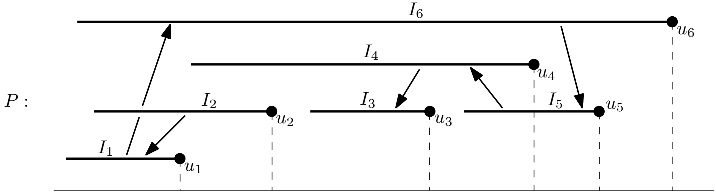


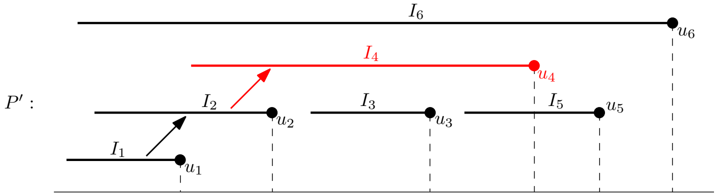


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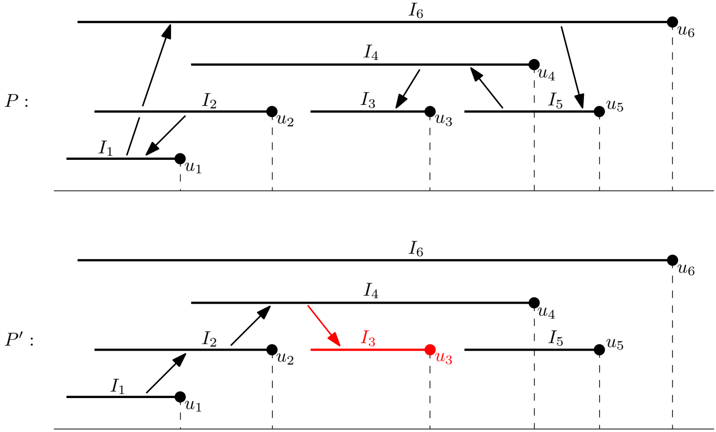


Normal path: $P' = (u_1, u_2, u_4, u_3, \quad , \quad )$

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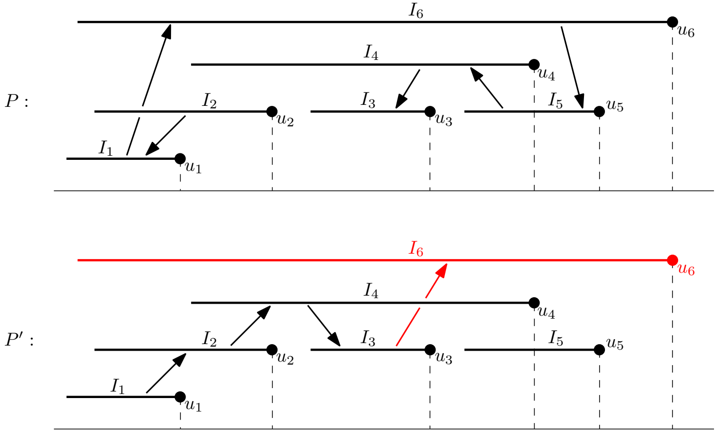


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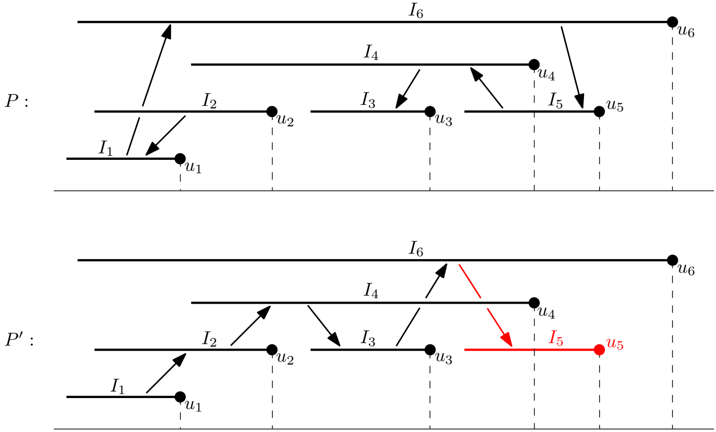
# Longest Path on Interval Graphs

## Normal paths in interval graphs

Example: path $P = (u_2, u_1, u_6, u_5, u_4, u_3)$




Normal path: $P' = (u_1, u_2, u_4, u_3, u_6, u_5)$

# Longest Path on Interval Graphs: Algorithm sketch

Given a proper interval deletion set $D$ of $G$, where $|D| = k$:

1. partition $G \setminus D$ into:
   - a collection of "reducible" sets and
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   - $O(n)$ such new intervals

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5. dynamic programming algorithm on $\widehat{G}$
   - compute in $O(\kappa^3 n) = O(k^9 n)$ time a max. weight path in $\widehat{G}$
   - this corresponds to a longest path of $G$

# Conclusions & Outlook

- "FPT inside P" offers an alternative way to deal with problems in P:
  - $f(k)$ can possibly become polynomial
  - a nice interplay with fast approximation algorithms, providing suitable parameters
  - one can aim at reducing "slow" polynomial running times (e.g. $O(n^3)$ or higher)
  - but also $O(n^2)$ (or less) for more practical applications

- `Longest Path on Interval Graphs`:
  - Can we significantly improve the running time of $O(k^9 n)$?

# Conclusions & Outlook

- Exploit the rich toolbox of "classical" FPT algorithms:
  - data reductions
  - kernelization
  - ...
- Lower bounds subject to established complexity conjectures
  - 3SUM
  - SETH
  - Boolean Matrix Multiplication
  - ...
- "FPT inside P" for big data / streaming
- Implementation / experiments of newly developed algorithms

Thank you for your attention!