# Erdős-Pósa property of planar $H$-minor models with prescribed vertex sets

## - Dichotomy theorem

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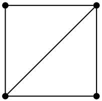
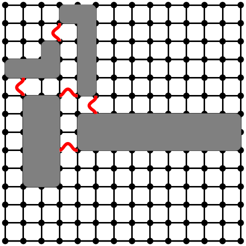
$H$-expansion ($H$-minor model) :



$H$



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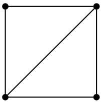

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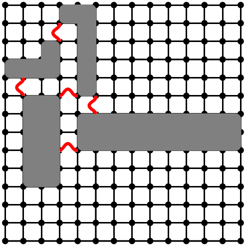

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$G$ has an $H$-expansion $\leftrightarrow$ $H$ is isomorphic to a minor of $G$.

A graph class $\mathcal{C}$ satisfies the **Erdős-Pósa** property if for every graph $G$ and an integer $k$, there exists a function $f(k, \mathcal{C})$ such that either

- $G$ contains $k$ pairwise vertex-disjoint subgraphs of $\mathcal{C}$, or
- there is a vertex set of size at most $f(k, \mathcal{C})$ that meets all subgraphs in $\mathcal{C}$.

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NO.

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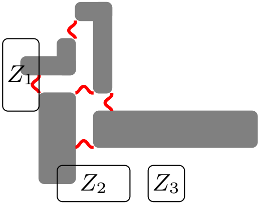
We would like to think about $H$-expansions containing some vertices of prescribed sets.



# $H$-expansions intersecting certain sets

Let $H$ be one vertex graph. Let $G$ be a graph, $S, T$ be disjoint vertex subsets of $G$. Then $H$-expansions intersecting both $S$ and $T$ satisfy the Erdős-Pósa property.

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Proof : Suppose there are no $k$ pairwise vertex-disjoint $H$-expansions intersecting both $S$ and $T$.
→ There are no $k$ pairwise vertex-disjoint paths from $S$ to $T$.
→ By Menger's theorem, there is a vertex set $A$ of size at most $k-1$ that meets all paths from $S$ to $T$.

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Let $H$ be one vertex graph. Let $G$ be a graph, $S_1, \ldots, S_m$ be disjoint vertex subsets of $G$. Then $H$-expansions intersecting at least two sets of $S_1, \ldots, S_m$ satisfy the Erdős-Pósa property.

Obtained from Mader's $\mathcal{S}$-path Theorem (78).

Let $G$ be a graph, and $Z_1, \ldots, Z_m$ be (not necessarily disjoint) vertex subsets of $G$, $\mathcal{Z} := \{Z_1, \ldots, Z_m\}$.

An $H$-expansion $F$ is called **$(\mathcal{Z}, \ell)$-intersecting** if there are $\ell$ distinct sets $Z$ of $\mathcal{Z}$ where $V(F) \cap Z \neq \emptyset$.

**Question :** Does the class of $(\mathcal{Z}, \ell)$-intersecting $H$-expansions have the Erdős-Pósa property?

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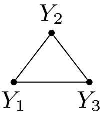

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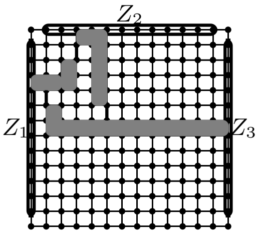

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There are no two pairwise vertex-disjoint $(\mathcal{Z}, 3)$-intersecting $H$-expansions.

**Theorem (Marx, K 15)**

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Our bound on the covering set significantly depends on the function in Grid Minor Theorem. By Chekuri and Chuzhoy (14), we have a polynomial bound in $k, \ell, |V(H)|$, and the bound does not depend on the number of given prescribed sets in $\mathcal{Z}$.

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Ingredients :

(1) slightly improvement of Rooted Grid Theorem (Marx, Seymour, Wollan 13).

(2) We prove the Erdős-Pósa property for pure $(\mathcal{Z}, \ell)$-intersecting $H$-expansions.

(3) Theorem by Burger (04) for bounded tree-width case.

(4) Reducing the original problem into pure expansions.

- We will show by induction on $\ell$. We assume that $H$ is planar. For easier discussion, we assume that $H$ consists of at least $\ell$ components.

1. Rooted Grid Expansion

Let $Z_1, \ldots, Z_m$ be (not necessarily disjoint) vertex subsets of $G$, $\mathcal{Z} := \{Z_1, \ldots, Z_m\}$.

A vertex set $W \subseteq V(G)$ is said to admit a $(\mathcal{Z}, k)$-**partition** if there is a partition $L_1, \ldots, L_x$ of $W$ and an injection $\gamma : \{1, \ldots, x\} \to \{1, \ldots, m\}$ such that

- $|L_i| \leqslant k$ and $L_i \subseteq Z_{\gamma(i)}$.

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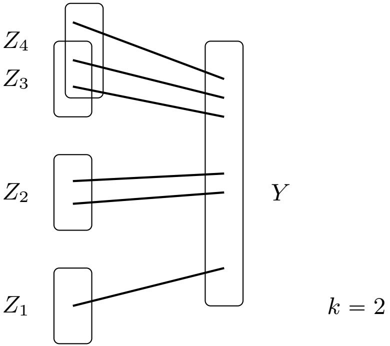
For a vertex set $Y$ of $G$, a set of pairwise vertex-disjoint $n$ paths from $\bigcup_{Z \in \mathcal{Z}} Z$ to $Y$ is called a $(\mathcal{Z}, Y, k)$**-linkage** of order $n$ if the end vertices in $\bigcup_{Z \in \mathcal{Z}} Z$ admit a $(\mathcal{Z}, k)$-partition.



## Lemma (Variant of Menger)

For $k, \ell$ and $Y \subseteq V(G)$, either $G$ contains

- a separation $(A, B)$ in $G$ of order less than $k\ell$ such that $Y \subseteq V(B)$ and $B \backslash V(A)$ contains at most $\ell - 1$ sets of $\mathcal{Z}$, or
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Proof Idea : Adding a set of $k$ vertices that are completely adjacent to $Z_i$ for each $i$.

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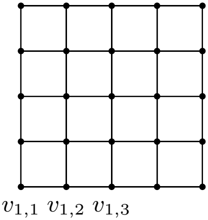
**A $(\mathcal{Z}, k, \ell)$-rooted grid expansion of order $g$** is a $\mathcal{G}_g$-expansion satisfying the property that for each $1 \leqslant i \leqslant k\ell$, $V(\eta(v_{1,i}))$ contains a vertex $w_i$ and $\{w_1, \ldots, w_{k\ell}\}$ admits a $(\mathcal{Z}, k)$-partition.



$\ell \cdot \mathcal{G}_h$ : $\ell$ disjoint union of copies of $\mathcal{G}_h$

## Lemma

Every $(\mathcal{Z}, k, \ell)$-rooted grid expansion of order $k\ell(h+1)+1$ contains $k$ pairwise vertex-disjoint $(\mathcal{Z}, \ell)$-intersecting $\ell \cdot \mathcal{G}_h$-expansions.

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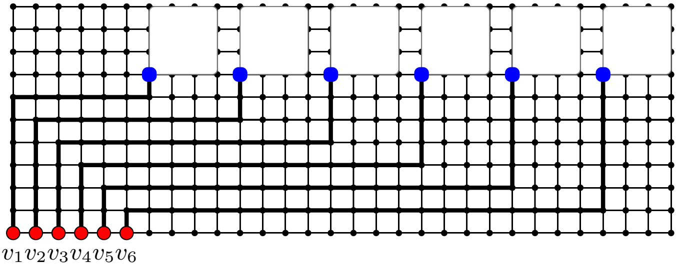


$k = 3, \ell = 2$

$Z_1 = \{v_1, v_3\}$, $Z_2 = \{v_6\}$, $Z_3 = \{v_4\}$, $Z_4 = \{v_2, v_5\}$

components of $H = H_1, H_2$

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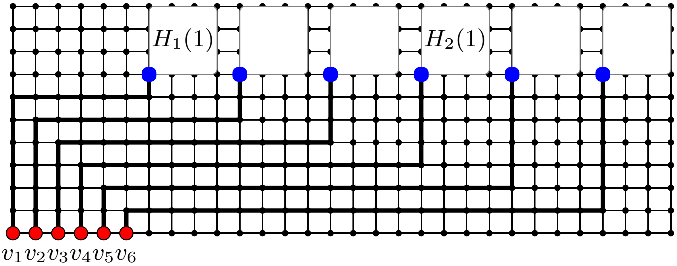


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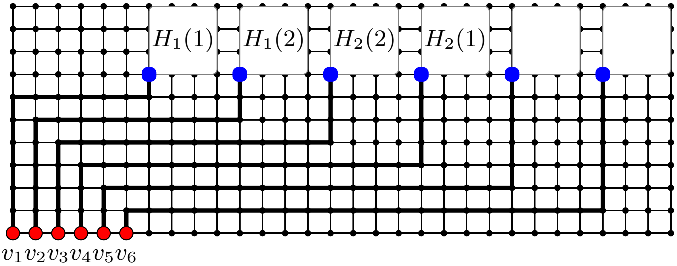


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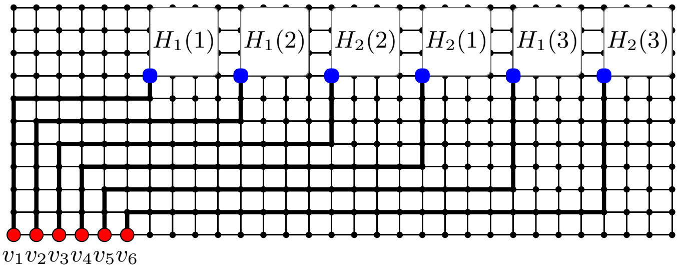


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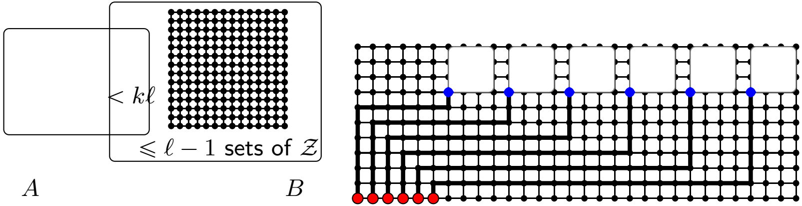
## Rooted Grid Theorem (Colorful version)

Let $g \geqslant k\ell$ and $n \geqslant g(k^2\ell^2 + 1) + k\ell$. If $G$ contains a $\mathcal{G}_n$-expansion, then either there is a separation $(A, B)$ of order less than $k\ell$ in $G$ such that

- $B \backslash V(A)$ contains at most $\ell - 1$ sets of $\mathcal{Z}$ and contains a $\mathcal{G}_{n-k\ell}$-expansion,

or there is a $(\mathcal{Z}, k, \ell)$-rooted grid expansion of order $g$.

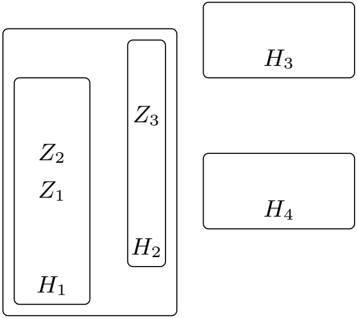
Original version was shown by Marx, Seymour, and Wollan (13).
One can find one of them in $poly(k \cdot \ell \cdot |V(G)|)$.



2. Pure $(\mathcal{Z}, \ell)$-intersecting $H$-expansions

A subgraph $F$ of a graph $G$ is called a **pure $(\mathcal{Z}, \ell)$-intersecting $H$-expansion** if there exist a subset $\mathcal{H} = \{H_1, \ldots, H_t\}$ of the set of components of $H$ and a graph $H'$ induced on the vertex set $\bigcup_{H_i \in \mathcal{H}} H_i$, and a model $\eta$ of $H'$ in $G$, and an injection $\alpha : \{1, \ldots, t\} \to 2^{\mathcal{Z}}$ such that

- $F$ is the image of $\eta$,
- each $\alpha(i)$ is non-empty and for $Z \in \alpha(i)$, $\eta(V(H_i)) \cap Z \neq \emptyset$,
- $\alpha(1), \ldots, \alpha(t)$ are pairwise disjoint, and
- $|\bigcup_{1 \leqslant i \leqslant t} \alpha(i)| = \ell$.




$\mathcal{Z} = \{Z_1, Z_2, Z_3\}$

components of $H = H_1, H_2, H_3, H_4$

Big Idea : First establish the Erdős-Pósa property of pure $(\mathcal{Z}, \ell)$-intersecting $H$-expansions. (Irrelavent vertex argument is possible)

Do not confuse :
no $k$ disjoint $(\mathcal{Z}, \ell)$-intersecting $H$-expansions $\nrightarrow$ no $k$ disjoint pure $(\mathcal{Z}, \ell)$-intersecting $H$-expansions.

## Theorem (E-P property for pure expansions)

For positive integers $k, \ell, h$ and a non-empty planar graph $H$ with $h$ vertices and at least $\ell - 1$ components, there exists a function $f_1(k, \ell, h)$ with the following property: Either

(1) $G$ contains $k$ pairwise vertex-disjoint pure $(\mathcal{Z}, \ell)$-intersecting $H$-expansions.

(2) There is a vertex subset $T$ of size at most $f_1(k, \ell, h)$ in $G$ such that $G \backslash T$ contains no pure $(\mathcal{Z}, \ell)$-intersecting $H$-expansions.

Proof : Assume $\ell = 2$ and suppose $G$ has a large grid expansion.

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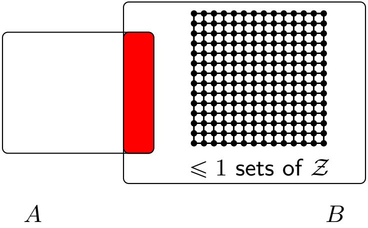
By Rooted Grid Theorem, $G$ has either a $(\mathcal{Z}, k, 2)$-rooted grid expansion, or a separation $(A, B)$ of order at most $2k$ where

- $B \backslash V(A)$ contains at most $1$ set of $\mathcal{Z}$ (say $\mathcal{Z}'$) and contains $\mathcal{G}_{g-2k}$-expansion



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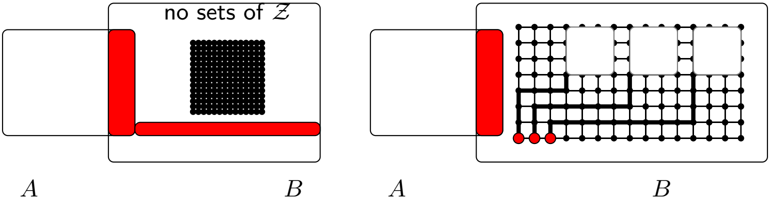
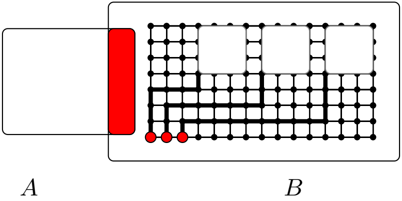
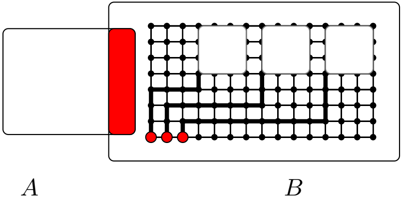
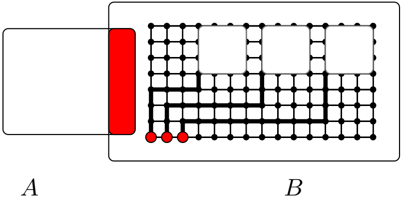



We apply Rooted Grid Theorem again on $B \backslash V(A)$ with $\mathcal{Z}'$.


$A$
$B$




In Case (1), $A \backslash V(B)$ contains no $k$ disjoint pure $(\mathcal{Z} \backslash \mathcal{Z}', 1)$-intersecting $H$-expansions. (Since $H$ has $\geqslant 2$ components, we can complete them using $(\mathcal{Z}', k, 1)$-rooted grid expansion in $B \backslash V(A)$.)

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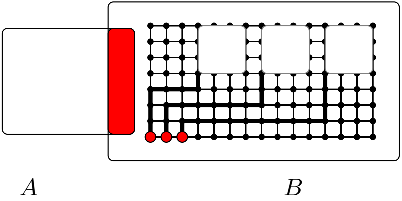


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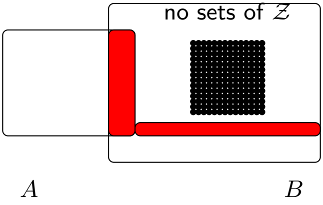
→ $T \cup V(A \cap B)$ is a vertex set which meets all pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions.

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$\rightarrow$ $T \cup V(A \cap B)$ is a vertex set which meets all pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions.

In Case (2), we can find an irrelavent vertex.
(* every component of pure expansions cannot be in $B \backslash V(A)$.)

Now suppose that $G$ admits a tree-decomposition of bounded width (say $w$).

$d$-subtree : subtree consisting of at most $d$ components

### Theorem (Burger 04)

Let $T$ be a tree and let $k, d$ be positive integers. Let $\mathcal{F}$ be a set of $d$-subtrees of $T$. Either

1. $T$ has $k$ pairwise vertex-disjoint subgraphs in $\mathcal{F}$.
2. There is a vertex subset $S$ of size at most $(d^2 - d + 1)(k - 1)$ such that $T \backslash S$ has no subgraphs in $\mathcal{F}$.

Use the fact that : all bags containing a vertex of a (pure) $H$-expansion induce an $h$-subtree in the tree-decomposition.

We have a $(w + 1)(h^2 - h + 1)(k - 1)$ bound on the covering set if $G$ has no $k$ pairwise vertex-disjoint (pure) $(\mathcal{Z}, \ell)$-intersecting $H$-expansions.

3. General $(\mathcal{Z}, \ell)$-intersecting $H$-expansions

## Theorem

For positive integers $k, \ell, h$ and a non-empty planar graph $H$ with $h$ vertices and at least $\ell - 1$ components, there exists a function $f(k, \ell, h)$ with the following property: Either

(1) $G$ contains $k$ pairwise vertex-disjoint $(\mathcal{Z}, \ell)$-intersecting $H$-expansions.

(2) There is a vertex subset $T$ of size at most $f(k, \ell, h)$ in $G$ such that $G \backslash T$ contains no $(\mathcal{Z}, \ell)$-intersecting $H$-expansions.

Unless $G$ has $k$ pairwise vertex-disjoint $(\mathcal{Z}, 2)$-intersecting $H$-expansions, we obtain

(1) a separation $(A, B)$ of order at most $2k$ where
   - $B \backslash V(A)$ contains $1$ set of $\mathcal{Z}$ (say, $\mathcal{Z}'$) and contains $(\mathcal{Z}', k, 1)$-rooted grid expansion of order $2k(14|V(H)| + 1)$, or

(2) a separation $(A, B)$ of order at most $4k$ where
   - $B \backslash V(A)$ contains no sets of $\mathcal{Z}$ and contains $\mathcal{G}_{g-4k}$-expansion.

## Theorem

For positive integers $k, \ell, h$ and a non-empty planar graph $H$ with $h$ vertices and at least $\ell - 1$ components, there exists a function $f(k, \ell, h)$ with the following property: Either

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Unless $G$ has $k$ pairwise vertex-disjoint $(\mathcal{Z}, 2)$-intersecting $H$-expansions, we obtain

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Case (2): $A \backslash V(B)$ contains no $k$ disjoint pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions which are not $H$-expansions. Also, $A \backslash V(B)$ contains no $k$ disjoint pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions which are $H$-expansions.

## Theorem

For positive integers $k, \ell, h$ and a non-empty planar graph $H$ with $h$ vertices and at least $\ell - 1$ components, there exists a function $f(k, \ell, h)$ with the following property: Either

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Case (2): $A \backslash V(B)$ contains no $k$ disjoint pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions which are not $H$-expansions. Also, $A \backslash V(B)$ contains no $k$ disjoint pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions which are $H$-expansions.

$\rightarrow$ $A \backslash V(B)$ contains no $2k - 1$ disjoint pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions. By Theorem for pure expansions, there is a vertex set $T$ that meets all pure $(\mathcal{Z}, 2)$-intersecting $H$-expansions. $T \cup V(A \cap B)$ gives a covering set.

# Conclusions

- The class of $(\mathcal{Z}, \ell)$-intersecting $H$-expansions has the Erdős-Pósa property iff $H$ is planar and it has at least $\ell - 1$ components.
- Is there an FPT algorithm parameterized by $k, \ell, |V(H)|$?
  If either $\ell = 1$ or the number of sets in $\mathcal{Z}$ is given as a constant, then yes.
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- Rooted $H$-expansions?
  (some specific vertex-model contains a vertex from some given set)
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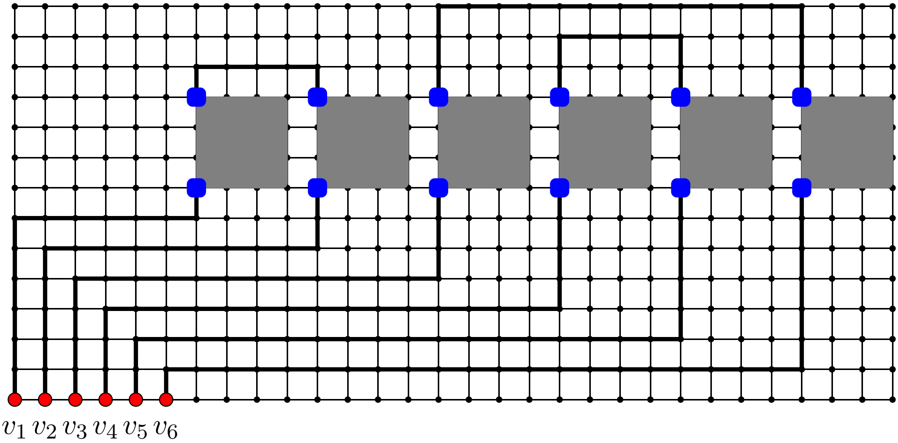
Thank you for your attention.


$v_1$ $v_2$ $v_3$ $v_4$ $v_5$ $v_6$

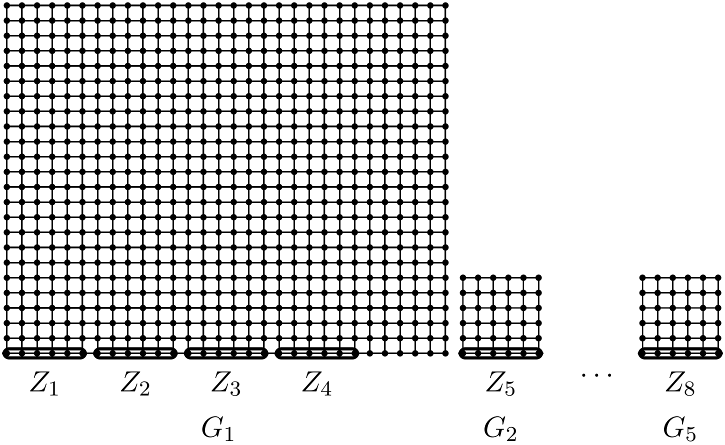



Figure: This is an example when $\ell = 8$ and the number of components of $H$ is $5$.