Enumeration of Minimal Connected Vertex Covers and Dominating Sets

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We show that a graph on n vertices has at most $O(1.8668^n)$ minimal connected vertex covers and these can be listed in time $O(1.8668^n)$.

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On graph classes

We show that graphs of chordality ≤ 5 have at most $O(1.6181^n)$ minimal connected vertex covers.

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On graph classes

We show that chordal graphs and distance hereditary graphs have at most $O(3^{n/3})$ minimal connected vertex covers.

Enumeration algorithms

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More clever branching algorithms

- Recursive algorithms with branching and reduction rules.
- Running time very often gives an upper bound on the number of objects that a graph can have.

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A connected vertex cover X is minimal if every $x \in X$ is either a cut vertex of $G[X]$ or it has a private edge.

Vertex covers in general

Observation

- *X is a vertex cover of $G = (V, E)$ if and only if $V \setminus X$ is an independent set of G.*

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- *If a vertex v is not in a vertex cover X then all neighbors of v are in X.*

A classical example

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The union of $n/3$ triangles has $3^{n/3}$ maximal independent sets and thus also $3^{n/3}$ minimal vertex covers.

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- The maximum number of minimal **connected** vertex covers of a graph or the enumeration of these has not been studied.
- Enumeration and maximum number of minimal vertex covers have been studied on graph classes. On triangle-free graphs there are better (tight) bounds.
- Computation of minimum sets is not a part of our motivation.

Minimal connected vertex covers of general graphs

Theorem

The maximum number of minimal connected vertex covers of an arbitrary graph is $O(1.8668^n)$*, and these can be enumerated in time* $O(1.8668^n)$*.*

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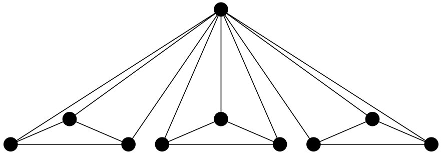

A graph with $3^{(n-1)/3} \approx 1.4422$ minimal connected vertex covers.

Sketch of the proof

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- set S of selected vertices
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Stop when: S is a minimal connected vertex cover or F is empty.

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At every step, we pick a vertex and we branch on the possibilities of **selecting** it to be placed in S and **discarding** it from being placed in minimal connected vertex covers containing S.

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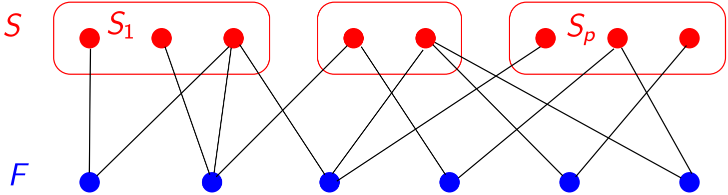

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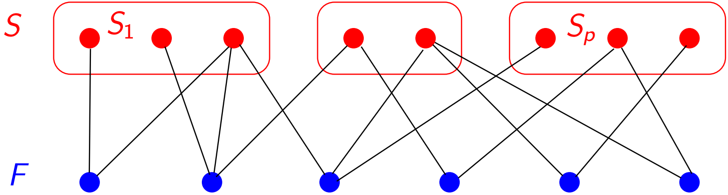

If F is an independent set then check every subset of F of size at most $p - 1$, and combine it with S to see if it gives a minimal connected vertex cover.

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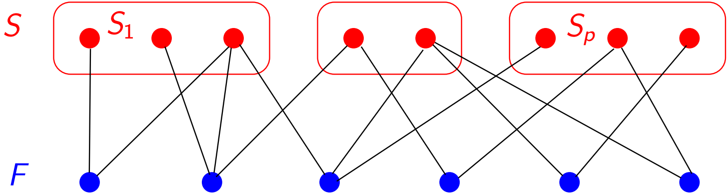

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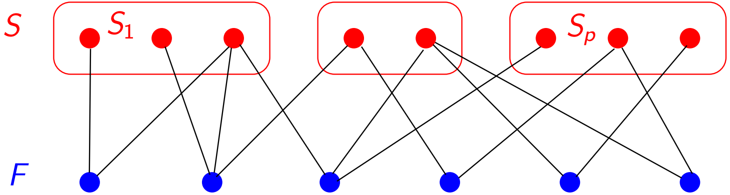

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$$T(n) = O(1.6181^{n-|F|} \cdot 2^{|F|}) = O(1.8668^n)$$

(Balancing at $|F| = 2n/3$)

Graphs of chordality at most 5

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Theorem

The maximum number of minimal connected vertex covers of a graph of chordality at most 5 is at most 1.6181^n, and these can be enumerated in time $O(1.6181^n)$.

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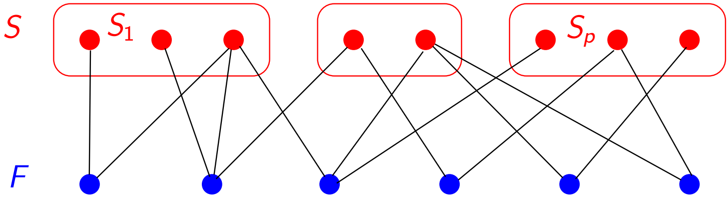

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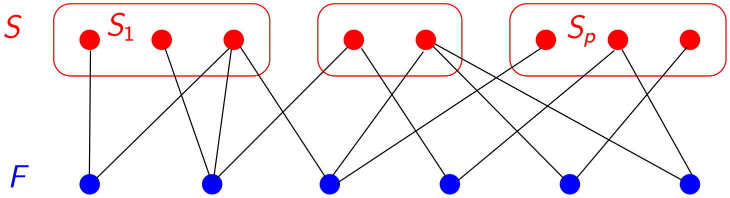

Recall that the running time of this part is dominated by $O(1.6181^n)$.

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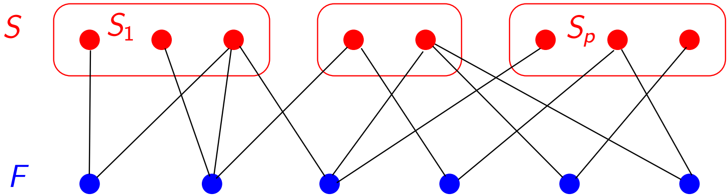

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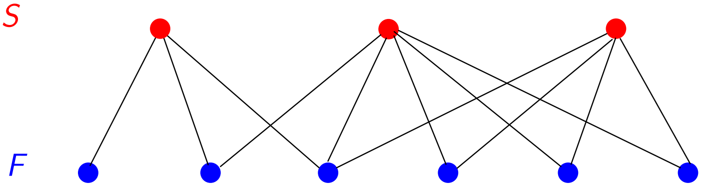

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- All edges with both endpoints in S are already covered.
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- We get a bipartite graph, hence of chordality at most 4. These are exactly the chordal bipartite graphs.
- Every chordal bipartite graph has a weakly simplicial vertex in each partite set.

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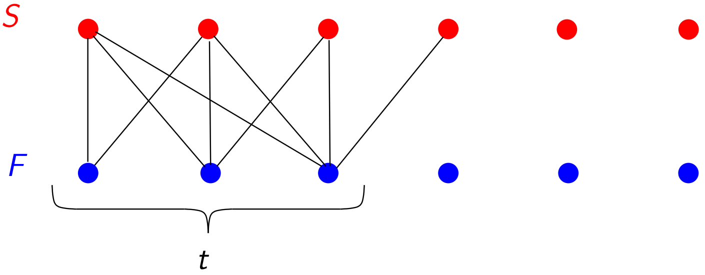

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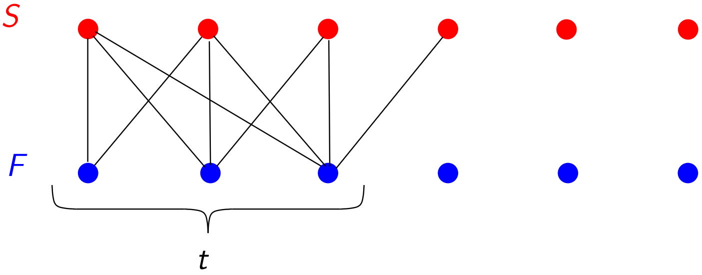

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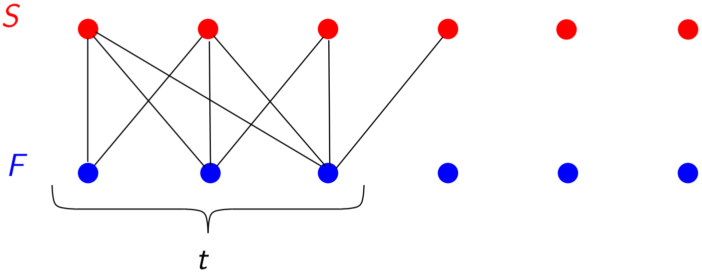

For a weakly simplicial vertex outside F, we branch on its neighbors. **Exactly one such neighbor can be selected.**

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Running time and maximum number of leaves:

$T(n) \leq t \cdot T(n - t)$

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Observe that the lower bound example has chordality at most 5.

Chordal graphs

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Theorem

The maximum number of minimal connected vertex covers of a chordal graph is at most $3^{n/3} < 1.4423^n$*, and these can be enumerated in time* $O(1.4423^n)$*.*

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S is a minimal vertex cover of G' if and only if $X = C \cup S$ is a minimal connected vertex cover of G.

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Lemma

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Consequence: Enumerating minimal connected vertex covers is equivalent to enumerating minimal vertex covers, which are at most $3^{n/3} < 1.4423^n$ in number and can be enumerated in time $O(1.4423^n)$.

Sketch of the proof of lemma

If $X = C \cup S$ is a minimal connected vertex cover of G then S is a minimal vertex cover of G'

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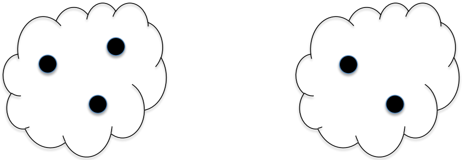
If $X = C \cup S$ is a minimal connected vertex cover of G then S is a minimal vertex cover of G' since exactly the edges incident to vertices of C are covered by the vertices of C.

If S is a minimal vertex cover of G' then $X = C \cup S$ is a vertex cover of G. We need to show that it is connected.

Sketch of the proof of lemma



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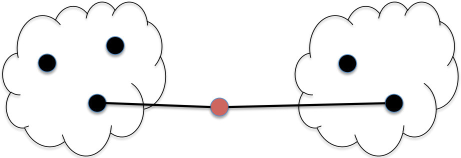

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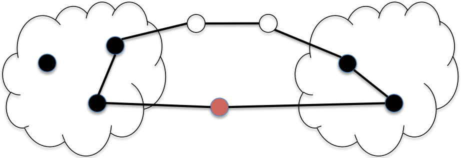

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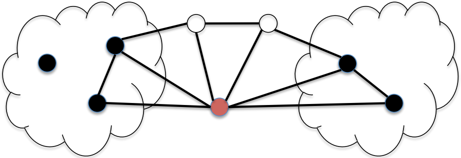

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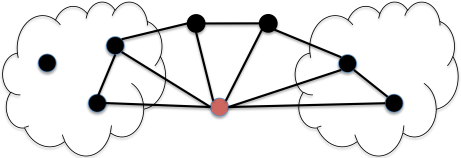

Distance hereditary graphs

A graph G is distance-hereditary if for every connected induced subgraph H of G, $\mathrm{dist}_H(u, v) = \mathrm{dist}_G(u, v)$ for $u, v \in V(H)$.

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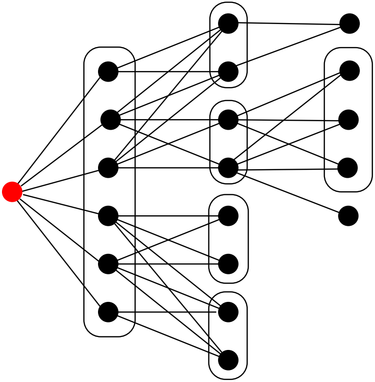
Theorem

The maximum number of minimal connected vertex covers of a distance-hereditary graph is at most $2 \cdot 3^{n/3}$, and these can be enumerated in time $O(1.4423^n)$.

Sketch of the proof



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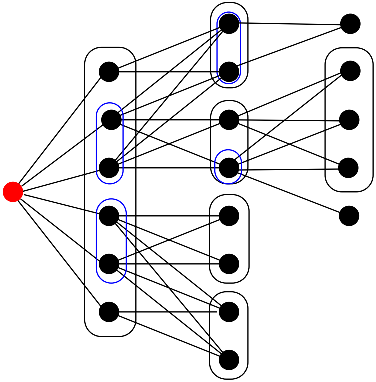

Split graphs and cobipartite graphs

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Proposition

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Split graphs and cobipartite graphs

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The number of minimal connected vertex covers of a split graph G is at most n, and these can be enumerated in time $O(n)$.

The number of minimal connected vertex covers of a cobipartite graph G is at most $n^2/4 + n$, and these can be enumerated in time $O(n^2)$.

Summary

Graph Class	Lower Bound	Upper Bound
general	$3^{(n-2)/3}$ $3^{(n-1)/3}$	2^n $O(1.8668^n)$
chordality≤ 5	$3^{(n-2)/3}$ $3^{(n-1)/3}$	2^n $O(1.6181^n)$
chordal	$3^{(n-2)/3}$ $3^{(n-1)/3}$	$O(1.7159^n)$ $3^{n/3}$
strongly chordal	$3^{(n-2)/3}$ $3^{(n-1)/3}$	$3^{n/3}$ $3^{n/3}$
split	1.3195^n n	1.3803^n n
cobipartite	1.3195^n $n^2/4$	1.3803^n $n^2/4+n$
interval	$3^{(n-2)/3}$ $3^{(n-1)/3}$	$3^{(n-2)/3}$ $3^{n/3}$
AT-free	$3^{(n-2)/3}$ $3^{(n-1)/3}$	$O^*(3^{n/3})$ $O(1.6181^n)$
distance-hereditary	$3^{(n-2)/3}$ $3^{(n-1)/3}$	$3^{n/3}\cdot n$ $2\cdot 3^{n/3}$
cographs	m $3^{(n-1)/3}$	m $2\cdot 3^{n/3}$

Minimal connected dominating sets and

minimal connected vertex covers

Open Questions : minimal connected vertex cover

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- Is it possible to improve the upper bound $O(1.3803^n)$ for split graphs?

Thank You!