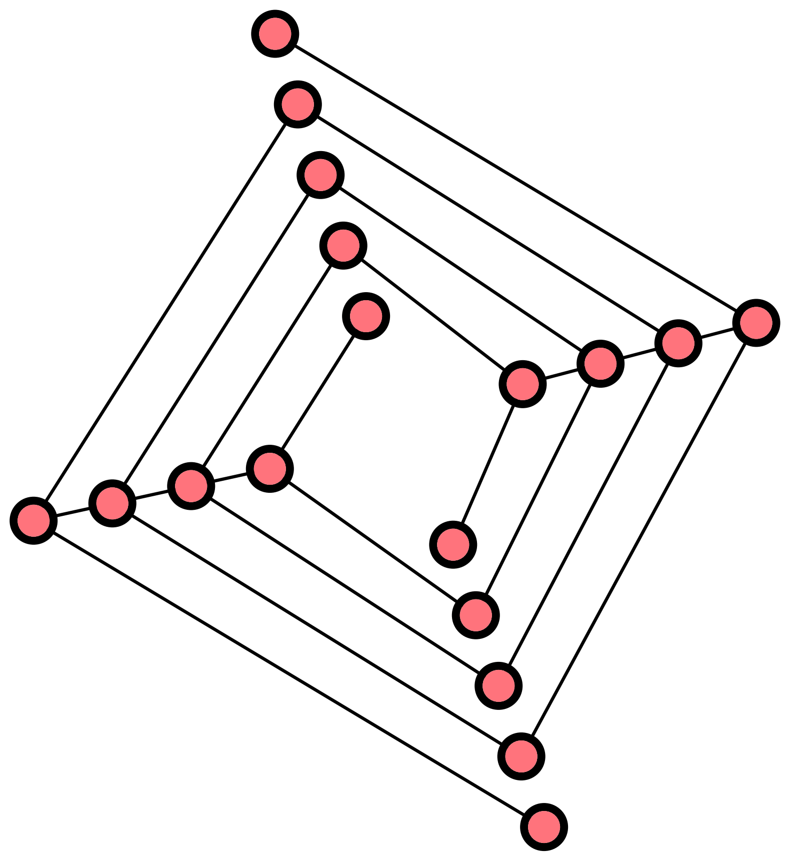
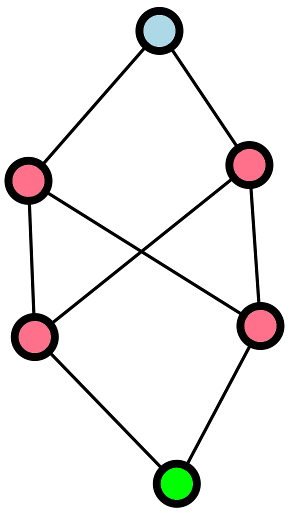
Sparsity and dimension

Gwenaël Joret
Université Libre de Bruxelles

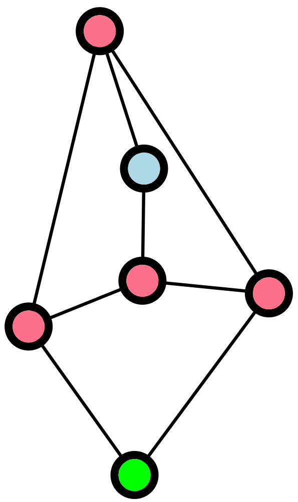
joint work with
Piotr Micek and Veit Wiechert



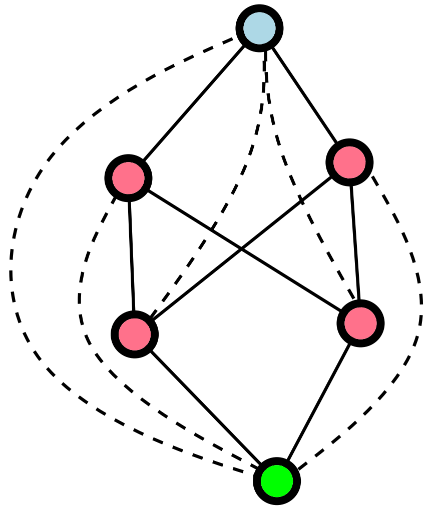
Drawing posets



diagram



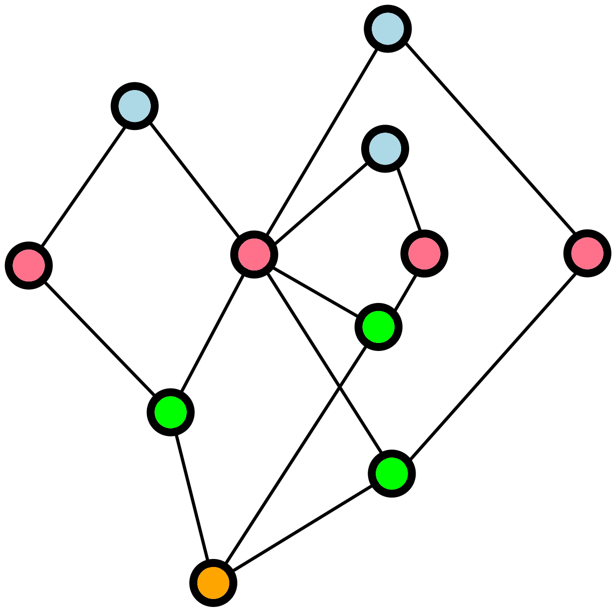
cover graph



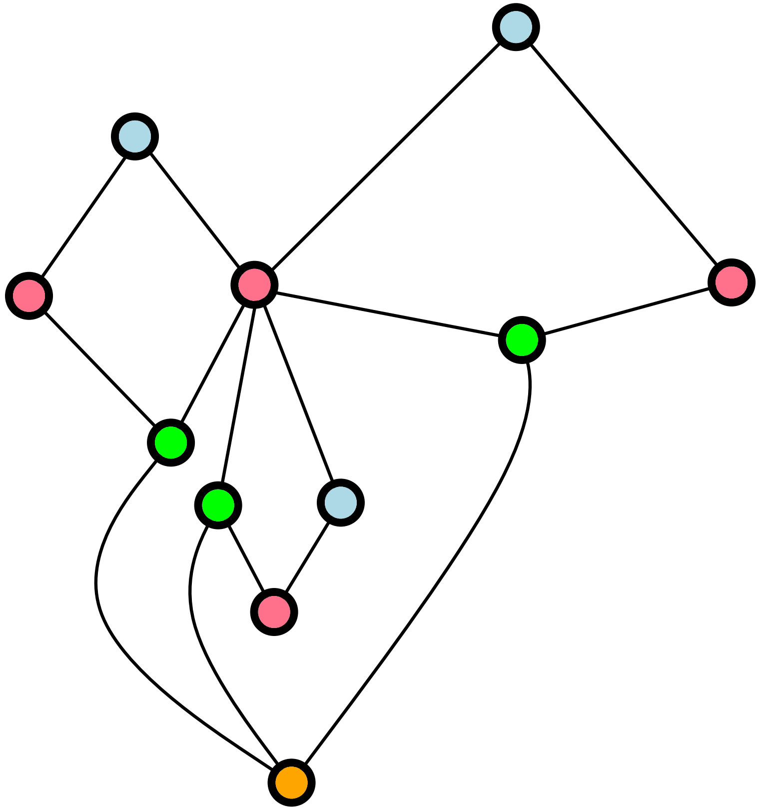
comparability graph

Drawing posets

poset with . . .

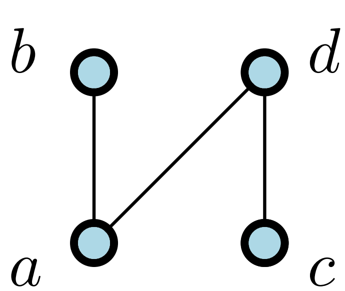


non-planar diagram

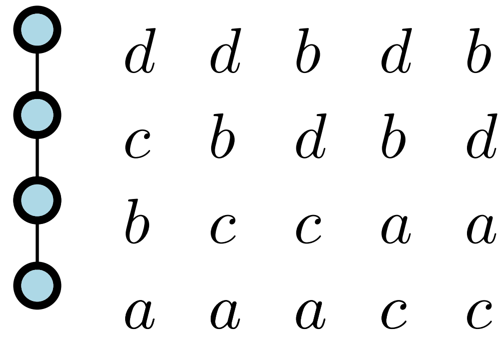


planar cover graph

Dimension



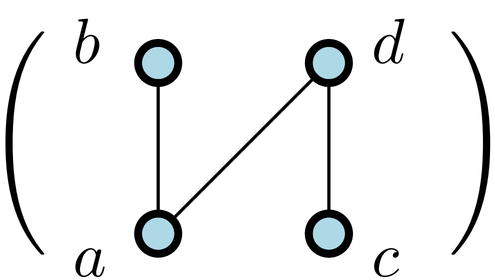
poset $\mathbf{P}$

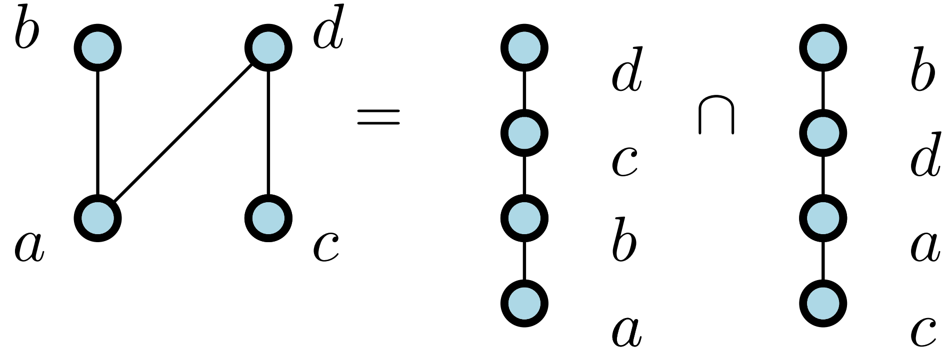


linear extensions of $\mathbf{P}$

Dimension of $\mathbf{P}$ is the minimum d such that there are d linear extensions $L_1, \ldots, L_d$ of $\mathbf{P}$ with

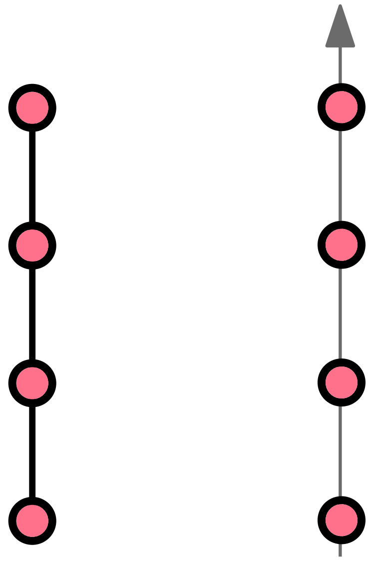
$$\mathbf{P} = \bigcap_{i \in [d]} L_i$$

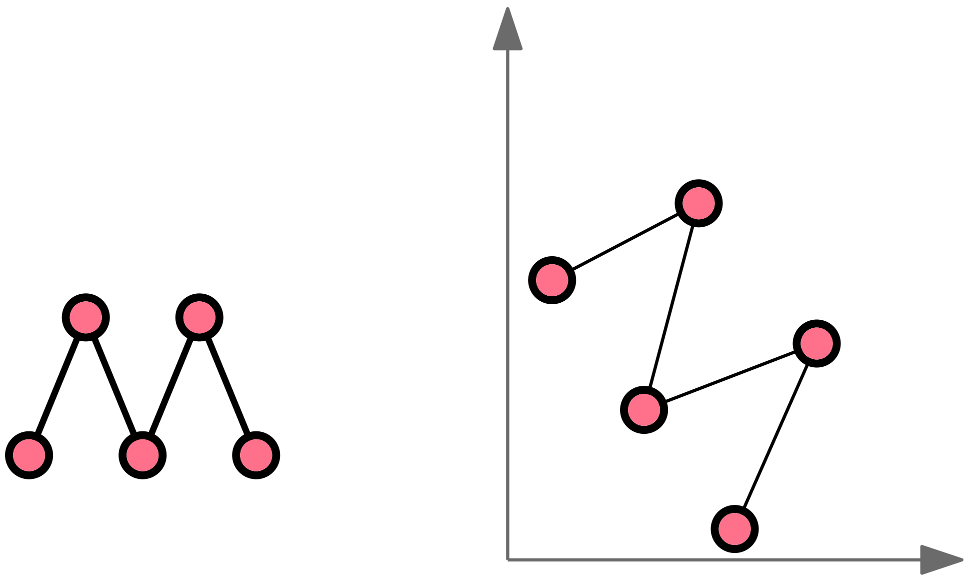
dim  $\leqslant 2$ as



Dimension: Geometric view

The dimension of a poset $\mathbf{P}$ is the least d such that $\mathbf{P}$ is isomorphic to a subposet of $\mathbb{R}^d$





Why dimension?

A natural notion...

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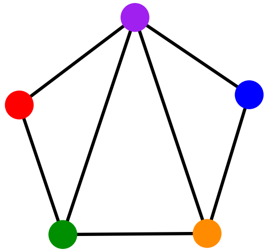
A natural notion...

...with interesting connections, e.g.:

Incidence posets:



G

$\mathbf{P}_G$

Why dimension?

A natural notion...

...with interesting connections, e.g.:

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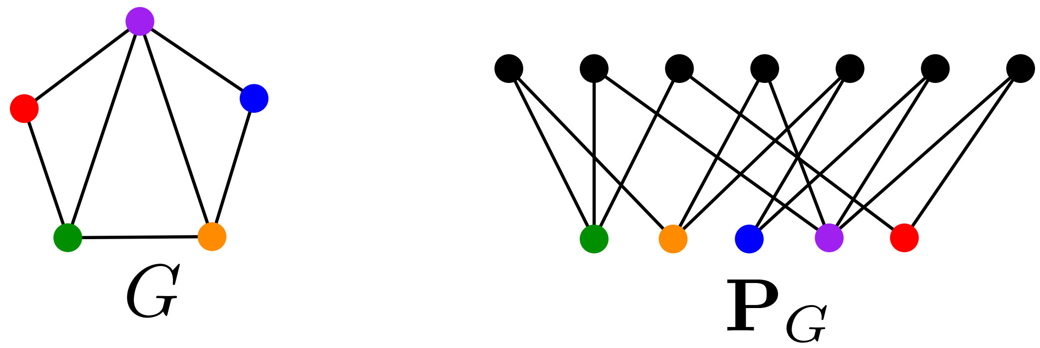
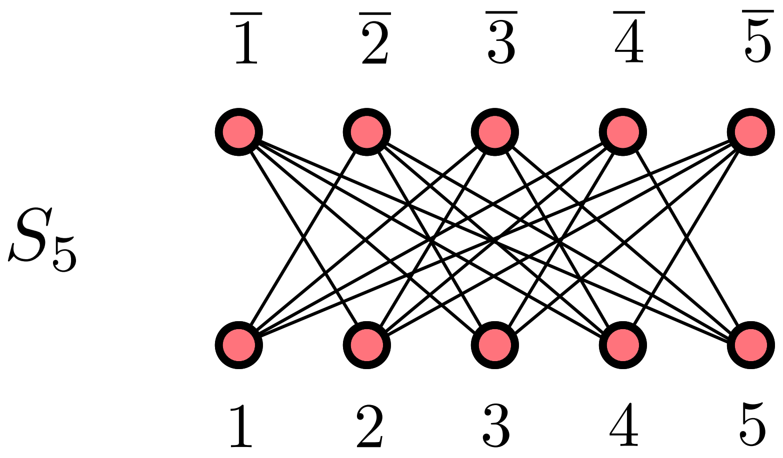


Schnyder '89

G planar $\Leftrightarrow \dim(\mathbf{P}_G) \leqslant 3$

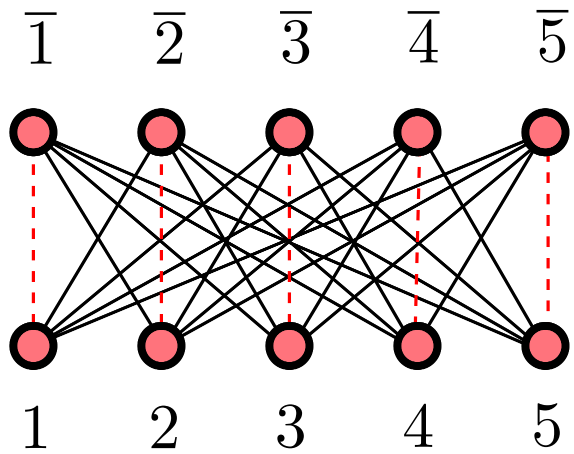
Standard examples have large dimension



$$\dim(S_n) = n$$

Standard examples have large dimension

S_5

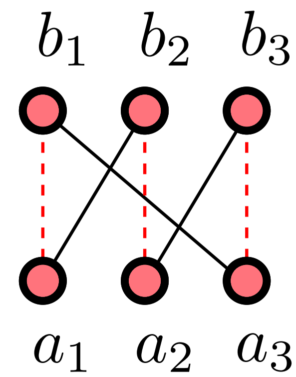


$$\dim(S_n) = n$$

Dimension: Hypergraph coloring problem

Dimension $=$ least number of linear extensions reversing all incomparable pairs (a, b)

Alternating cycle: Incomparable pairs $(a_1, b_1), \ldots, (a_k, b_k)$ s.t. $a_i \leqslant_P b_{i+1}$ $\forall i$ (cyclically)



Lemma: Set I of incomparable pairs can be reversed with one linear extension $\Leftrightarrow$ I has no alternating cycle

Hypergraph $\mathcal{H}$:

- vertex set $=$ $\{$ incomparable pairs $\}$
- hyperedges $\leftrightarrow$ alternating cycles
- $\chi(\mathcal{H}) = \dim(\mathbf{P})$

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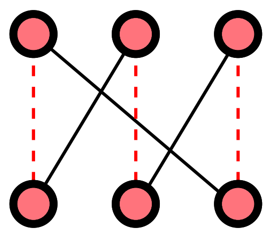

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Hypergraph $\mathcal{H}$:

- vertex set $=$ $\{$ incomparable pairs $\}$
- hyperedges $\leftrightarrow$ alternating cycles
- $\chi(\mathcal{H}) = \dim(\mathbf{P})$
- cliques $\leftrightarrow$ standard examples

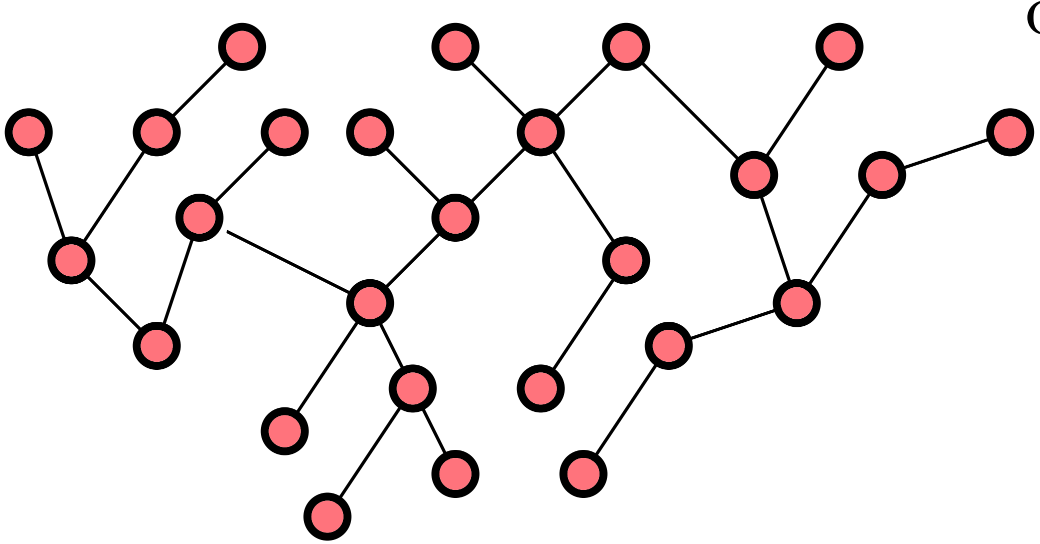
What is it going to be about?

"If a poset is nice then its dimension is small"

Trotter & Moore '77

If cover graph of $\mathbf{P}$ is a forest then

$\dim(\mathbf{P}) \leqslant 3$



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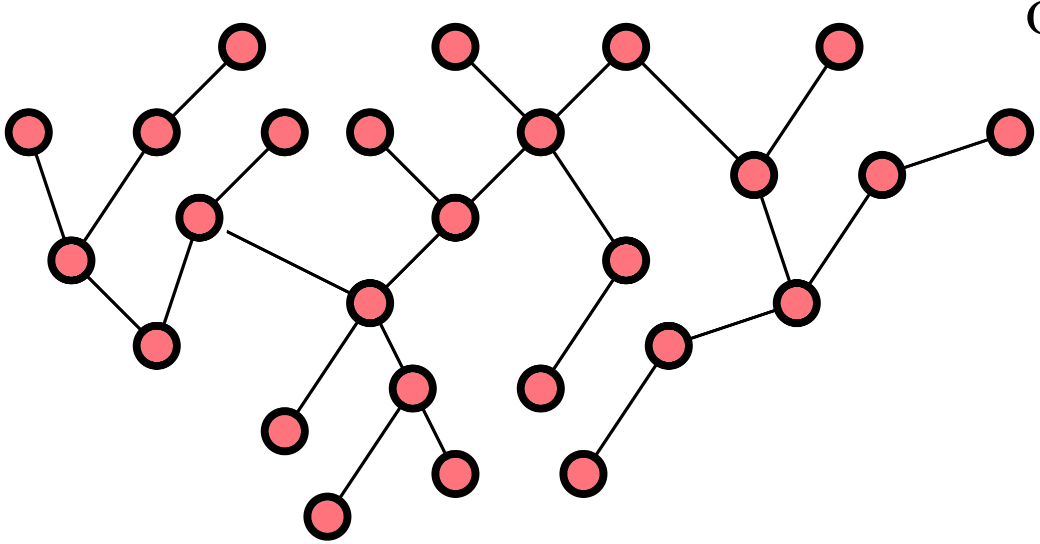


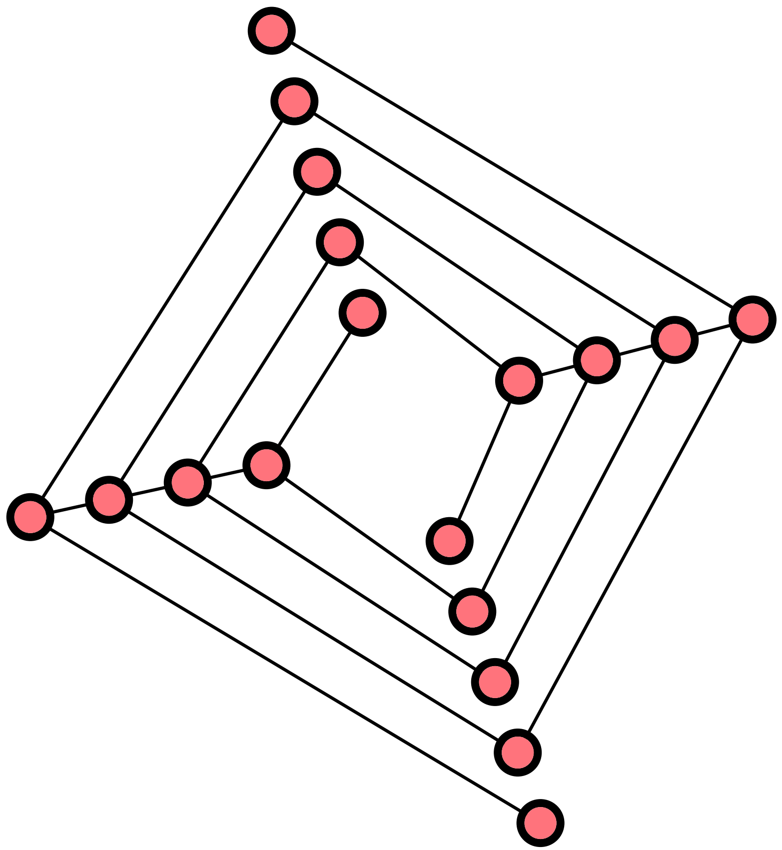
J., Micek, Trotter, Wang, Wiechert '14

If cover graph of $\mathbf{P}$ has treewidth $\leqslant 2$ then

$$\dim(\mathbf{P}) \leqslant 1276$$

Kelly's example

Kelly '81



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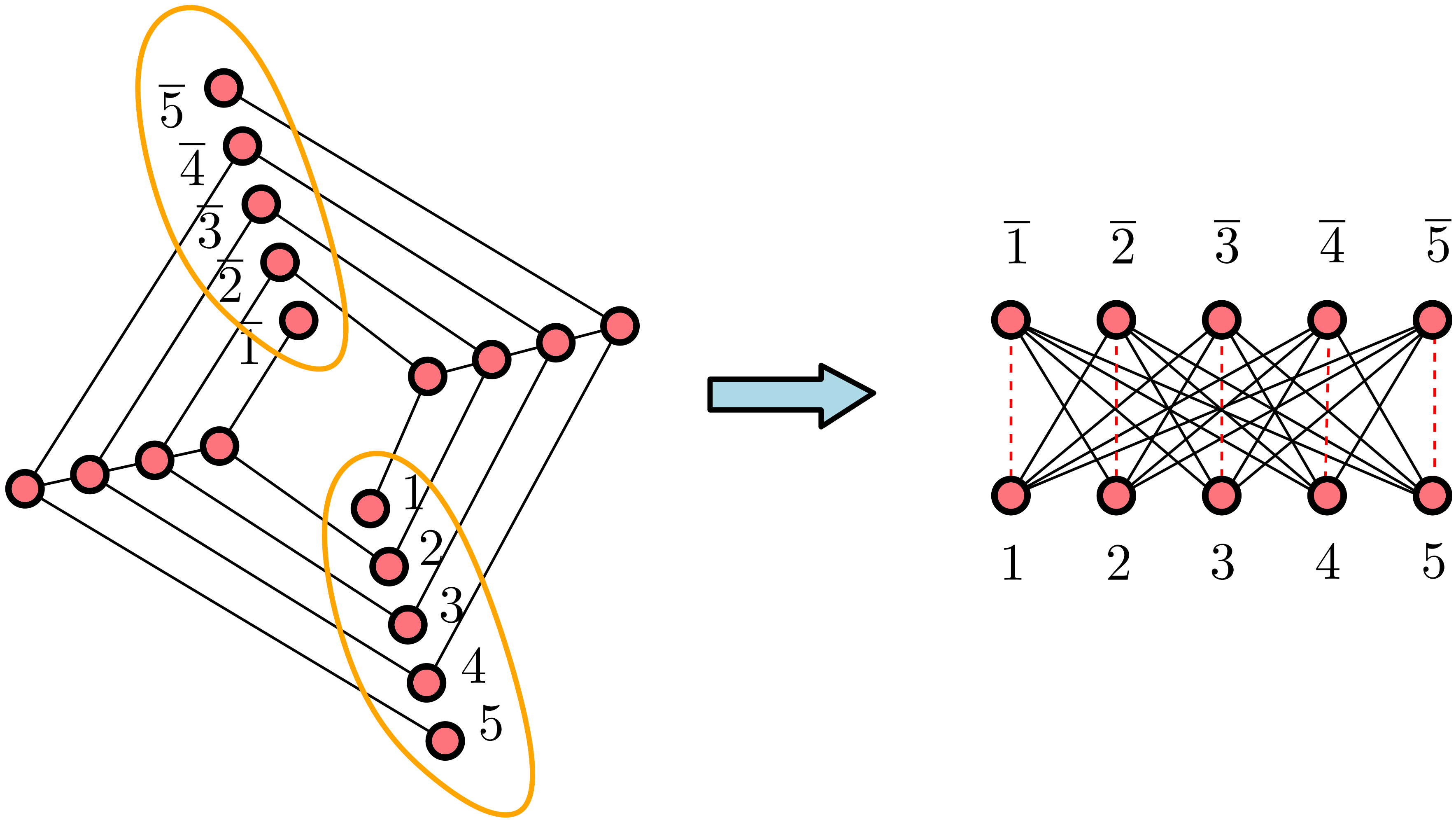
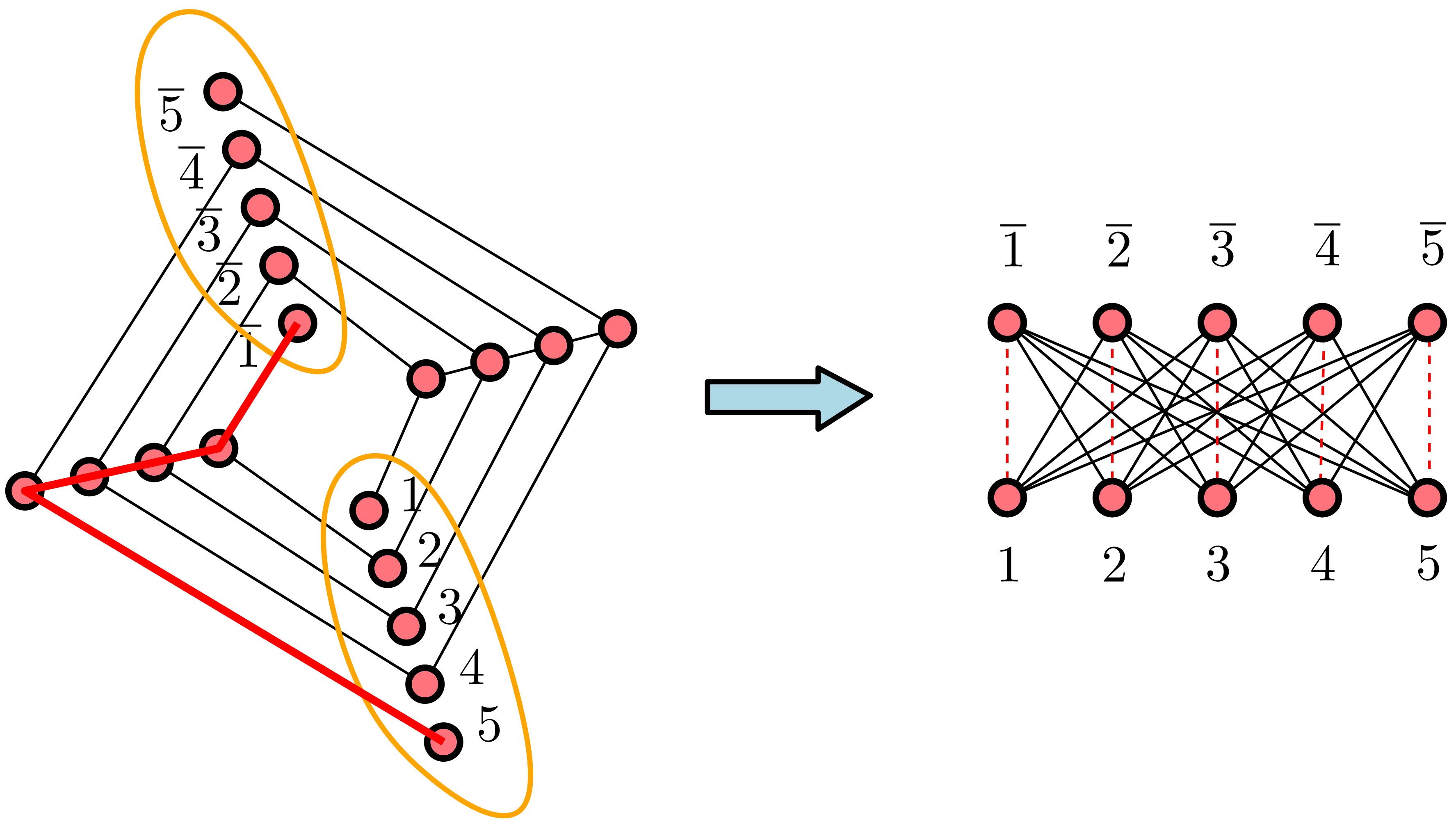


planar posets with arbitrarily large dimension

(cover graphs have pathwidth 3)

Kelly's example

Kelly '81



planar posets with arbitrarily large dimension

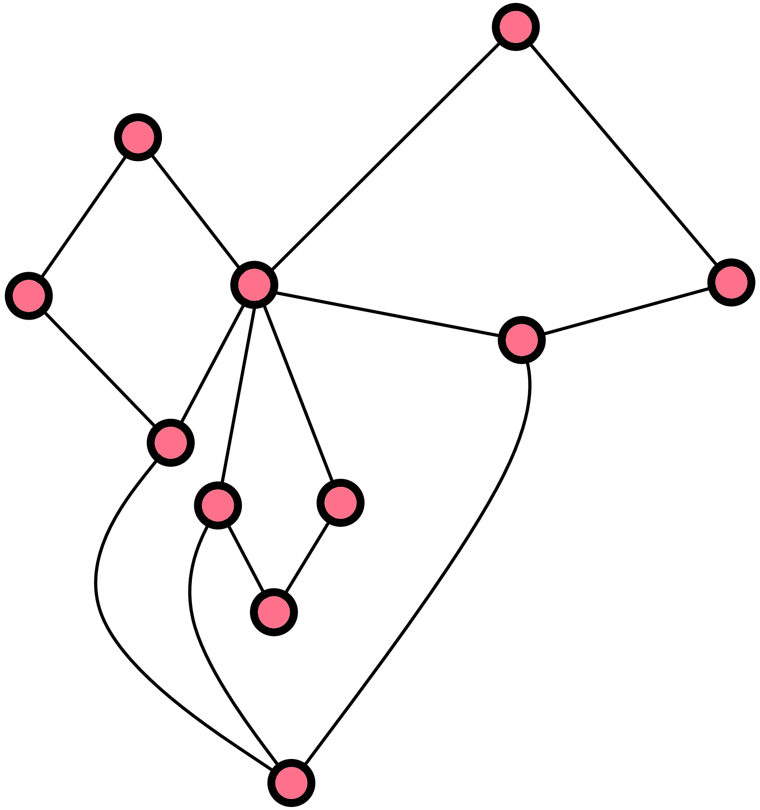
(cover graphs have pathwidth 3)

... but height unbounded!

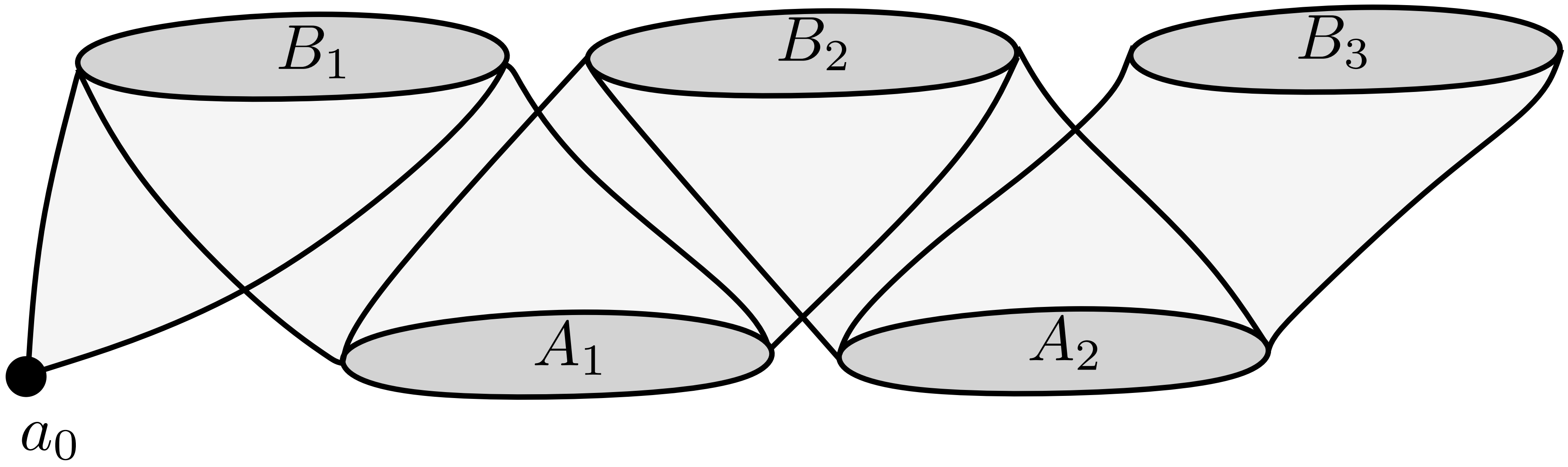
Planarity

Streib & Trotter '12

If $\mathbf{P}$ has height h and cover graph of $\mathbf{P}$ is planar then $\dim(\mathbf{P}) \leqslant c_h$

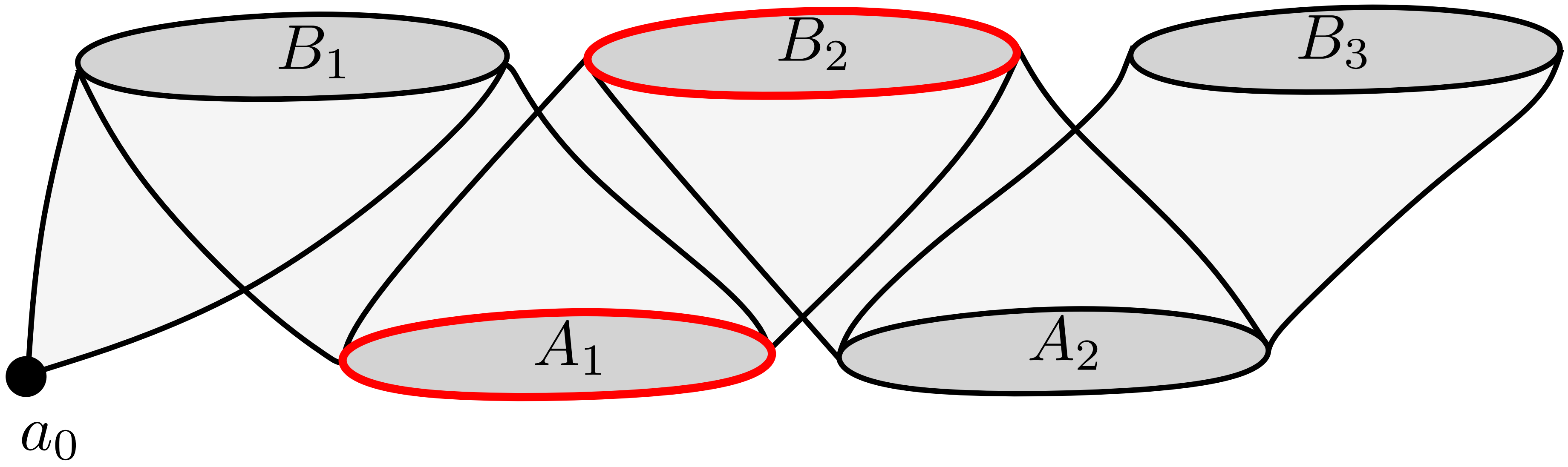


Unrolling



Lemma: $\exists i$ s.t. $\dim(A_i, B_i) \geqslant \dim(\mathbf{P})/2$ or $\dim(A_i, B_{i+1}) \geqslant \dim(\mathbf{P})/2$

Unrolling



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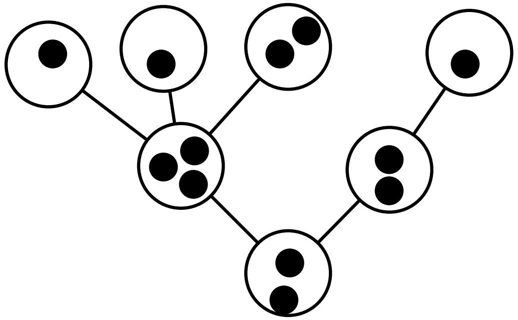
Treewidth, genus, minors, ...

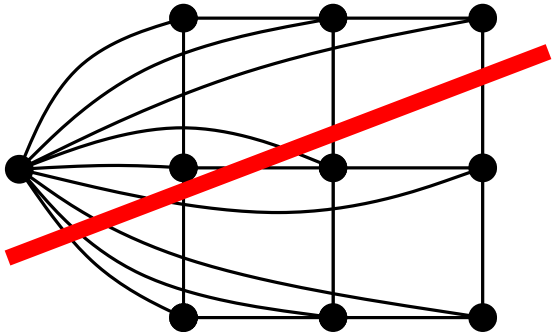
J., Micek, Milans, Trotter, Walczak, Wang '13

If $\mathbf{P}$ has height h and cover graph of $\mathbf{P}$ has treewidth t then $\dim(\mathbf{P}) \leqslant c_{h,t}$



Corollary using unrolling trick:

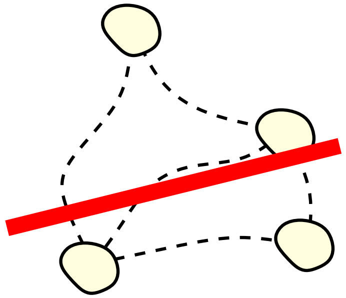
If $\mathbf{P}$ has height h and cover graph of $\mathbf{P}$ excludes an apex graph A as minor then $\dim(\mathbf{P}) \leqslant c_{h,A}$



Treewidth, genus, minors, ...

Walczak '14

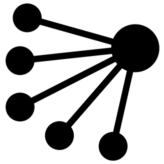
If $\mathbf{P}$ has height h and cover graph of $\mathbf{P}$ excludes a graph J as topological minor then $\dim(\mathbf{P}) \leqslant c_{h,J}$



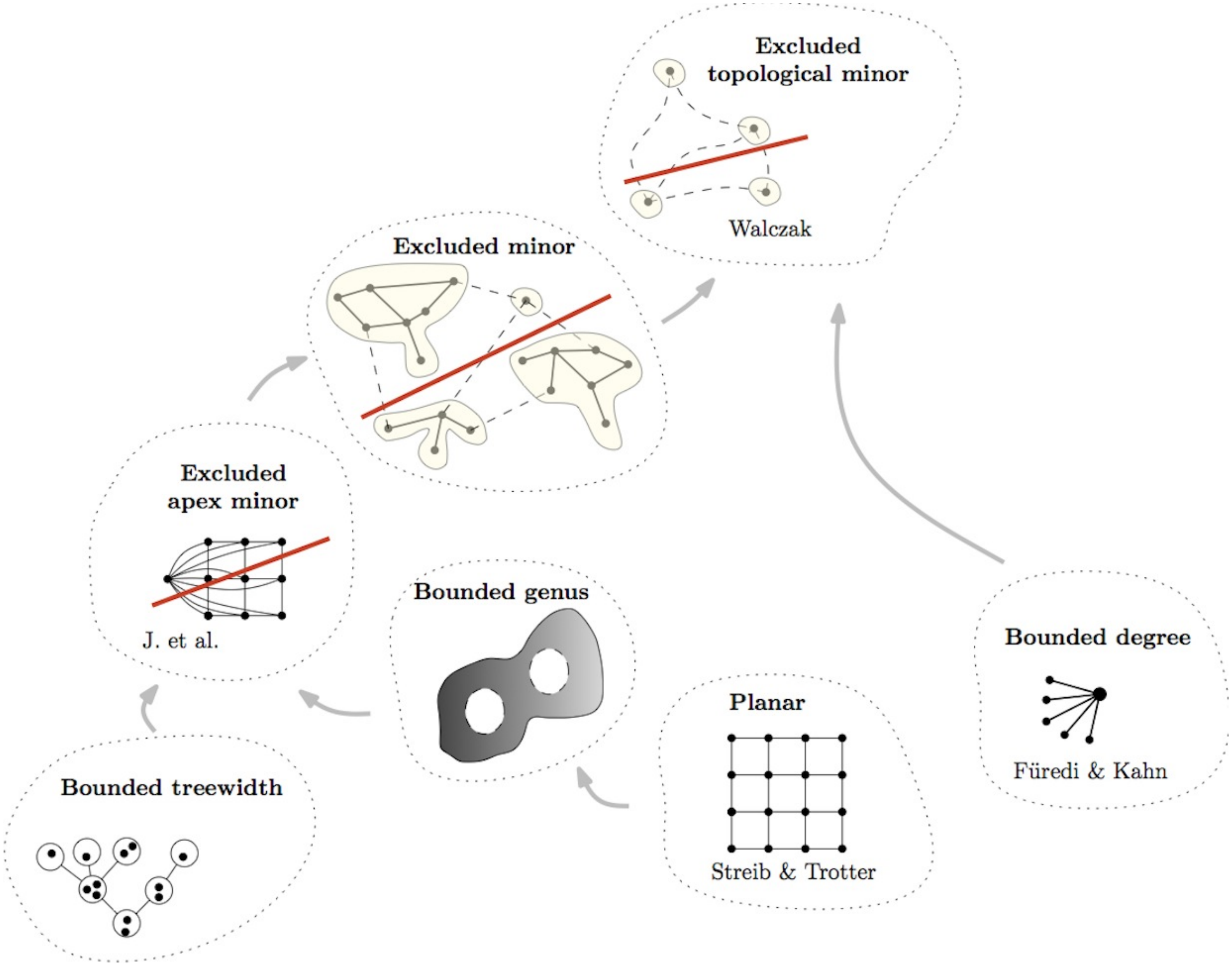
Uses:

Füredi & Kahn '86

If $\mathbf{P}$ has height h and cover graph of $\mathbf{P}$ has maximum degree Δ then $\dim(\mathbf{P}) \leqslant c_{h,\Delta}$

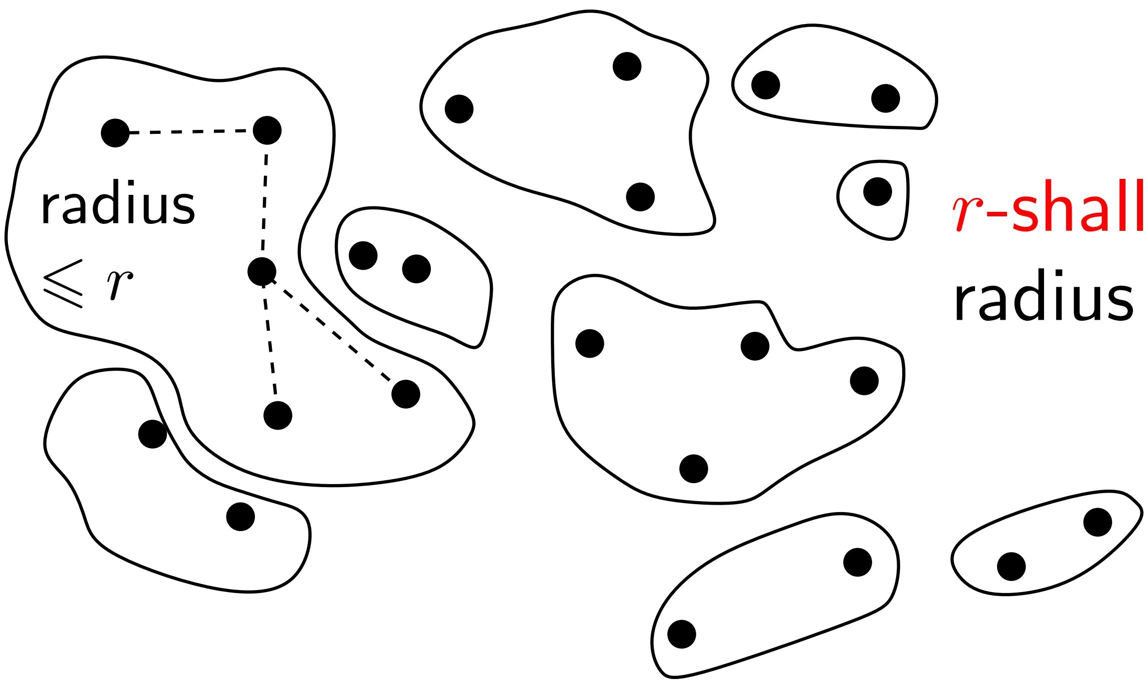


Picture so far



Bounded expansion

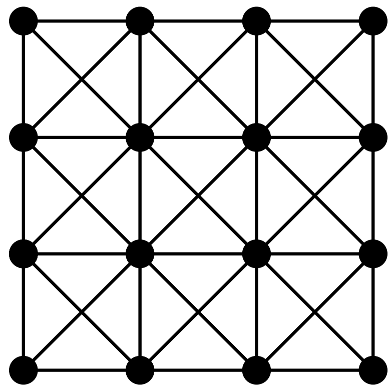
Class $\mathcal{G}$ has bounded expansion $\Leftrightarrow$ $\forall r$, all r-shallow minors of $G \in \mathcal{G}$ have average degree $\leqslant f(r)$



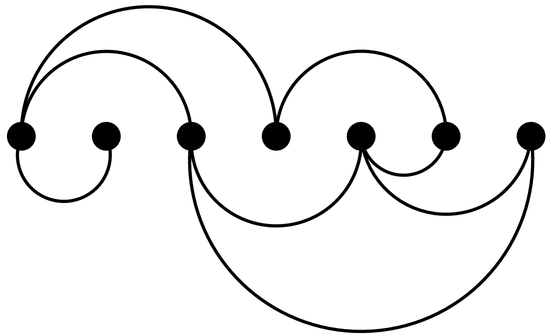
r-shallow minor: branch sets have radius $\leqslant r$

Bounded expansion: Examples from graph drawing

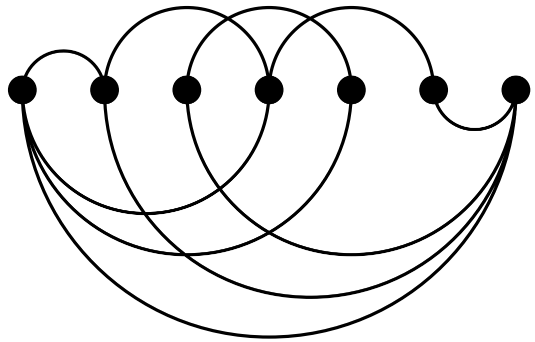
k-planar graphs



Bounded book thickness



Bounded queue number

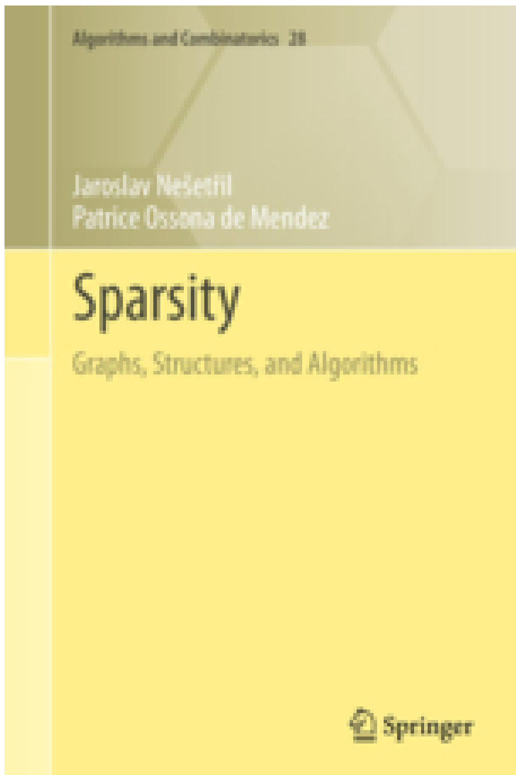


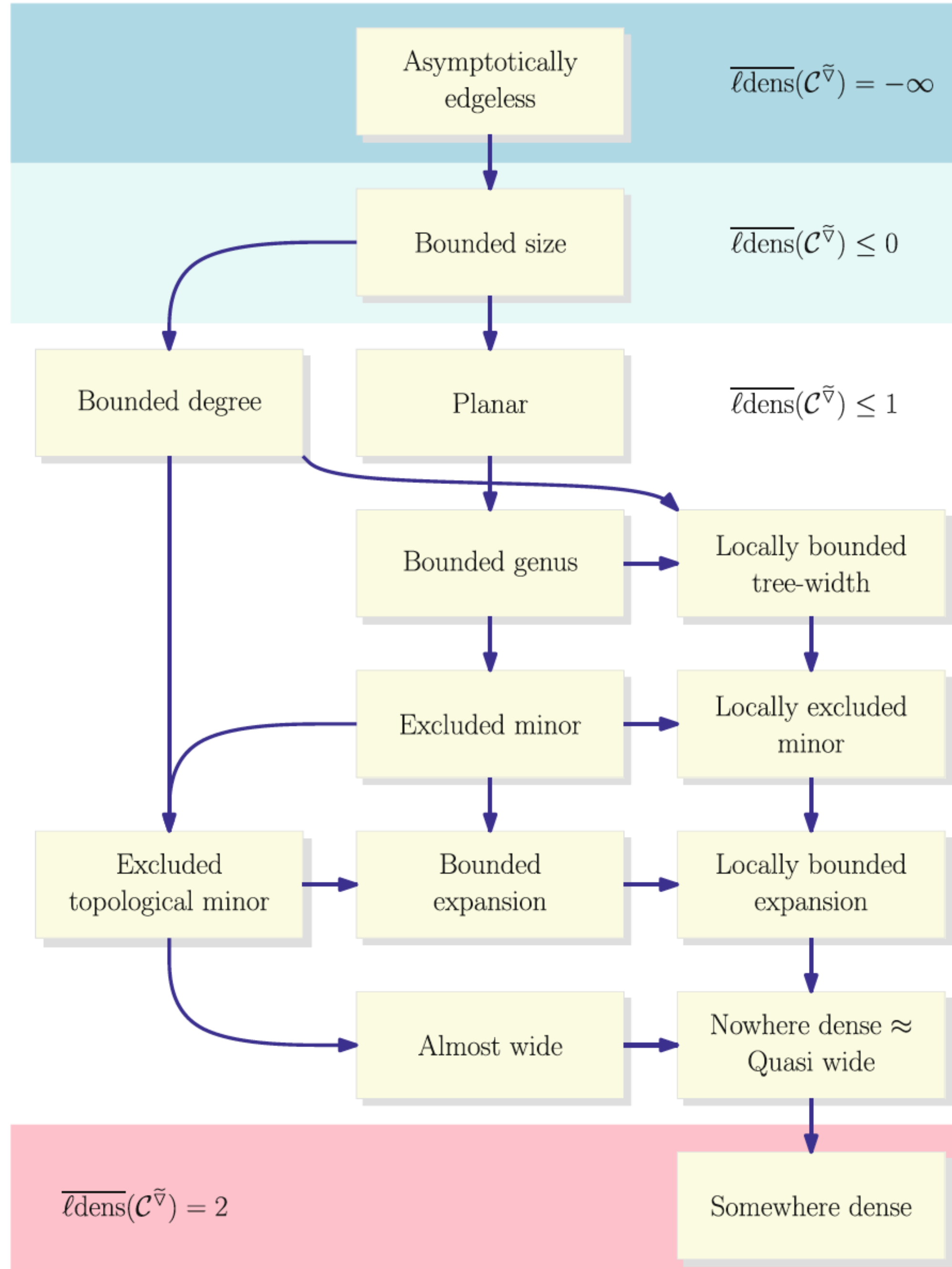
Bounded expansion

J., Micek, Wiechert '15

If $\mathbf{P}$ has height h and cover graph of $\mathbf{P}$ belongs to a class $\mathcal{G}$ with bounded expansion then $\dim(\mathbf{P}) \leqslant c_{h,\mathcal{G}}$

Beyond bounded expansion?

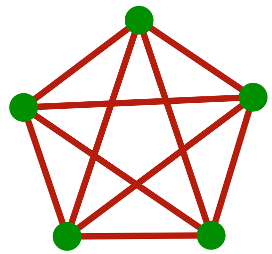




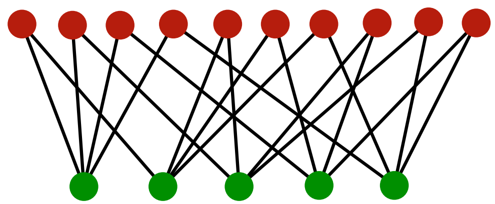
Bounded degeneracy?

Not enough

Incidence posets:



G



$\mathbf{P}_G$

Cover graph is 2-degenerate

Füredi, Hajnal, Rödl, Trotter '92

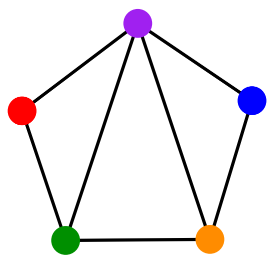
$\dim(\mathbf{P}_{\mathbf{K_n}}) = \log\log n + (\frac{1}{2} + o(1)) \log\log\log n$

Nowhere dense?

Class $\mathcal{G}$ is nowhere dense $\Leftrightarrow$ $\forall r\ \exists H$ s.t. H not r-shallow minor of any $G \in \mathcal{G}$

Not enough

Adjacency posets:



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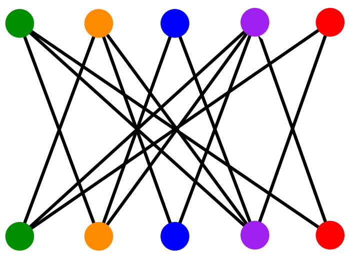


$\mathbf{AP}_G$

Lemma: $\dim(\mathbf{AP_G}) \geqslant \chi(G)$

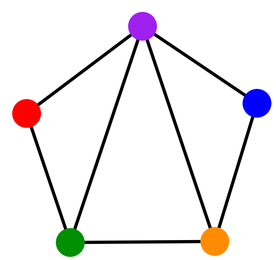
$\mathcal{G} = \{\text{graphs } G \text{ with } \Delta(G) \leqslant girth(G)\}$

- $\mathcal{G}$ nowhere dense, unbounded χ
  $\Rightarrow$ $\dim(\mathbf{AP_G})$ for $G \in \mathcal{G}$ unbounded
- $\forall G \in \mathcal{G}$, cover graph of $\mathbf{AP_G}$ also in $\mathcal{G}$

Locally bounded treewidth?

Not enough

Adjacency posets:



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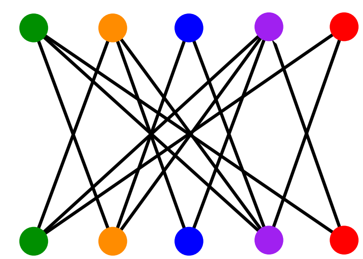


$\mathbf{AP}_G$

Lemma: $\dim(\mathbf{AP_G}) \geqslant \chi(G)$

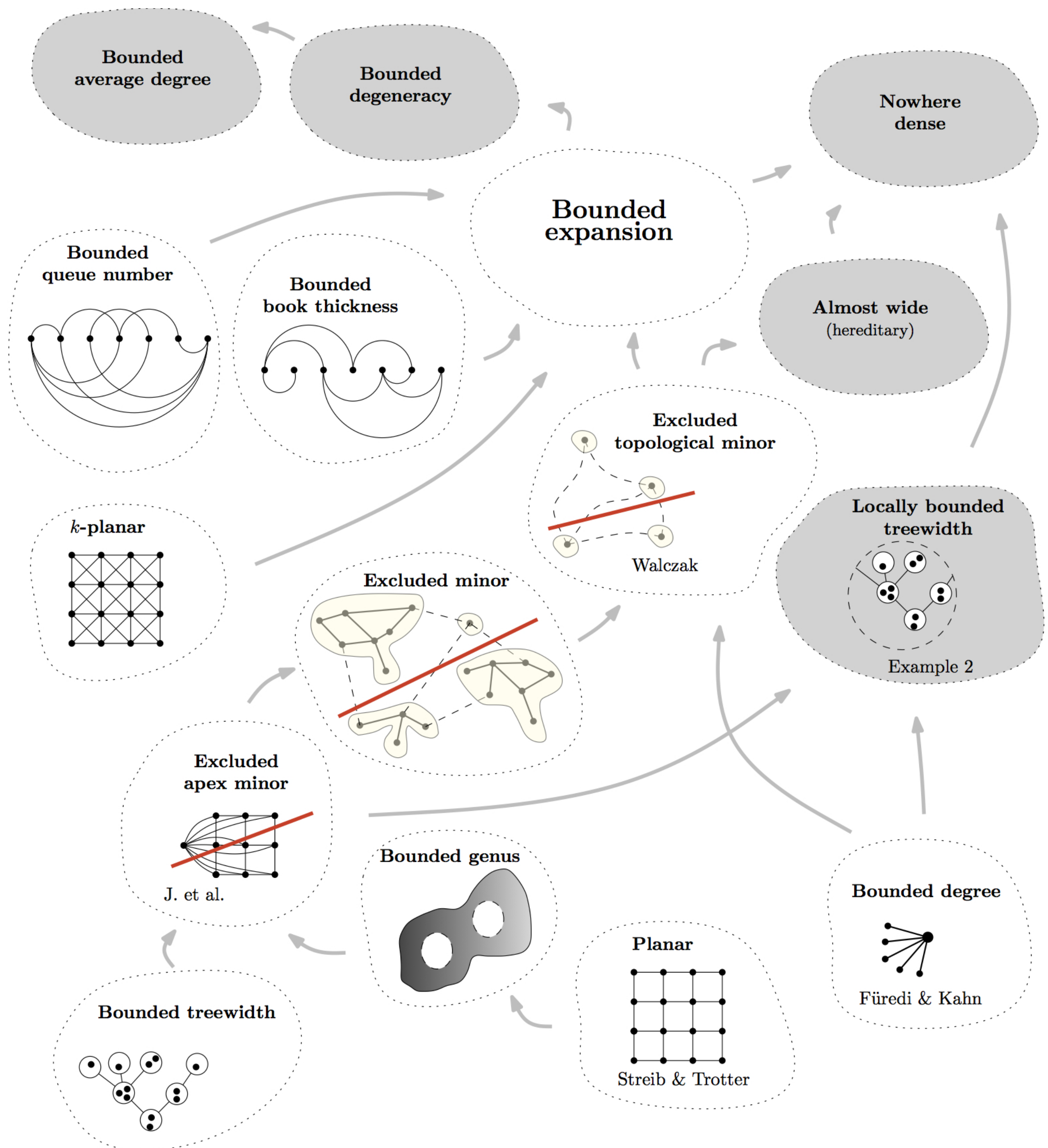
$\mathcal{G} = \{$graphs G with $\Delta(G) \leqslant \text{girth}(G)\}$

- $\mathcal{G}$ has locally bounded treewidth, unbounded χ

$\Rightarrow$ $\dim(\mathbf{AP_G})$ for $G \in \mathcal{G}$ unbounded

- $\forall G \in \mathcal{G}$, cover graph of $\mathbf{AP_G}$ also in $\mathcal{G}$

Updated picture



Bounded expansion: Proof

Class $\mathcal{G}$ has bounded expansion $\Leftrightarrow$ $\forall r$, all r-shallow minors of $G \in \mathcal{G}$ have average degree $\leqslant f(r)$

Proof ingredient 1

Class $\mathcal{G}$ has low tree-depth colorings if $\exists f$ s.t. $\forall G \in \mathcal{G}$, $\forall p$, there is an $f(p)$-vertex coloring of G s.t. union of every i color classes has tree-depth $\leqslant i$, $\forall i \leqslant p$

Nešetřil & Ossona de Mendez '08

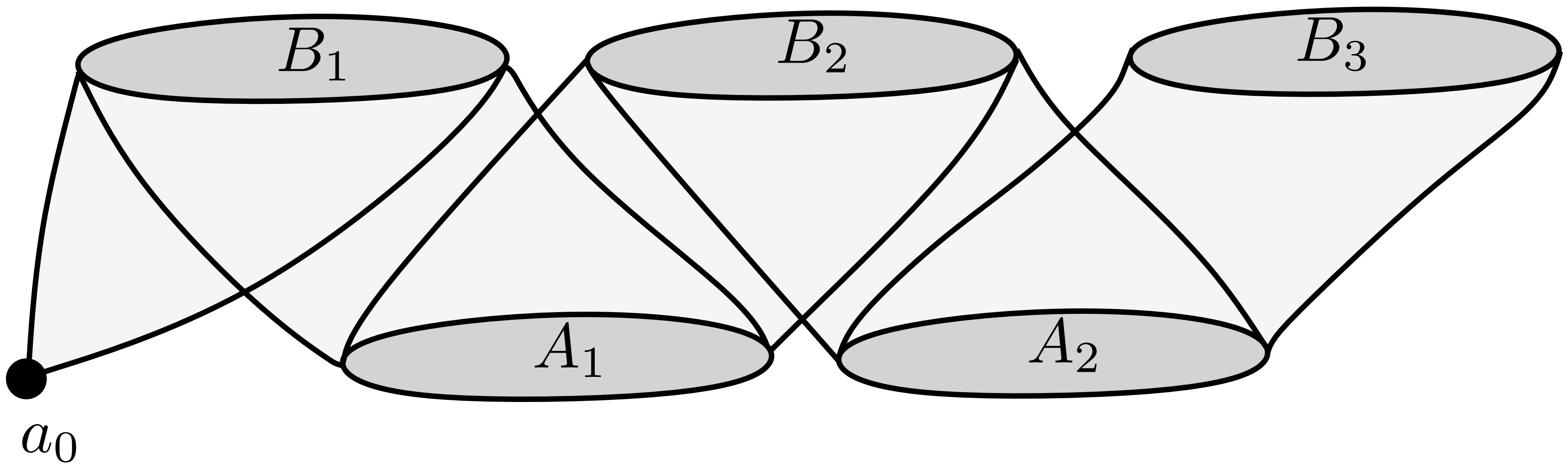
Class $\mathcal{G}$ has bounded expansion $\Leftrightarrow$ $\mathcal{G}$ has low tree-depth colorings

In the proof: such a coloring for $p = 2h - 1$

Bounded expansion: Proof

Fix low tree-depth coloring ϕ of cover graph with $\leqslant f(2h-1)$ colors

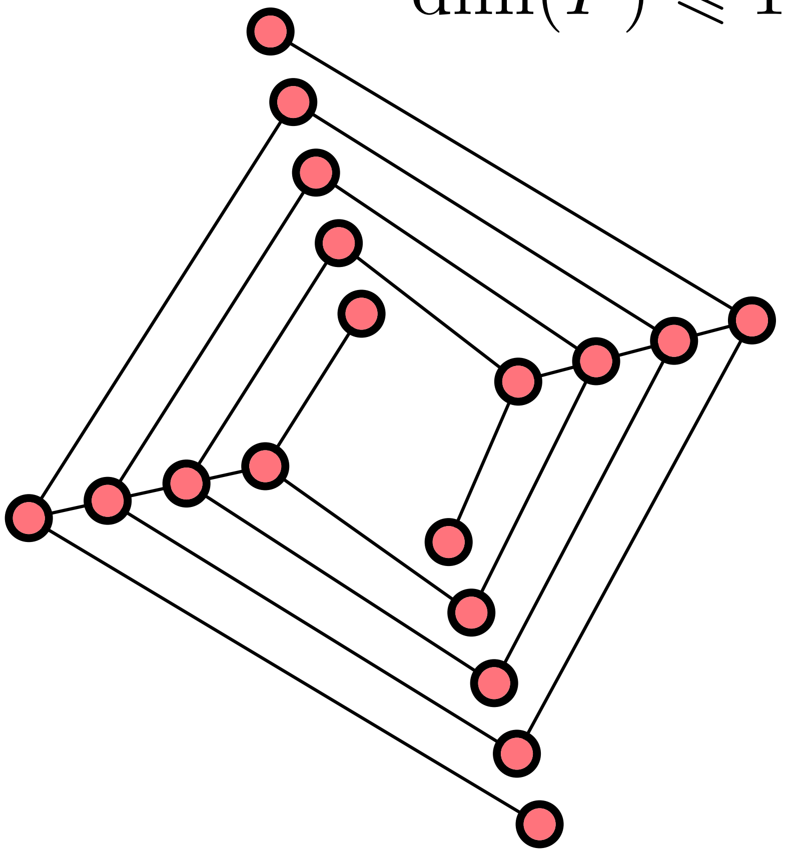
Proof ingredient 2
Iterated unrolling



Recall: there is a heavy block (A_i, B_i) or (A_i, B_{i+1}) with dimension $\geqslant \dim(\mathbf{P})/2$

Research directions

⋆ improve the bounds: e.g. (recent result)
$\dim(P) \leqslant 192h$, for planar P of height h



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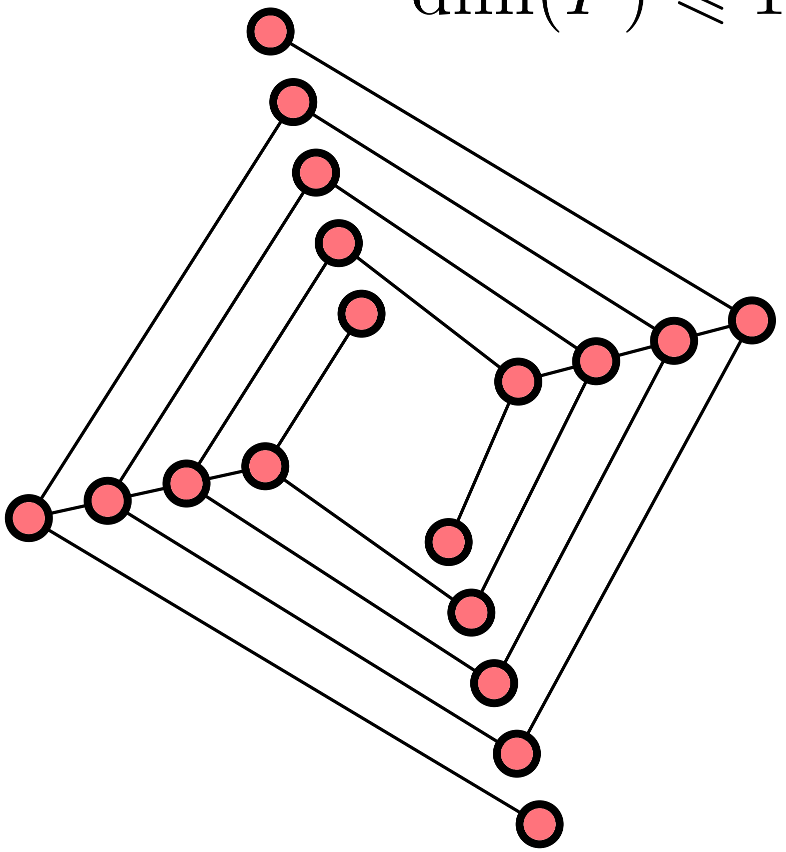

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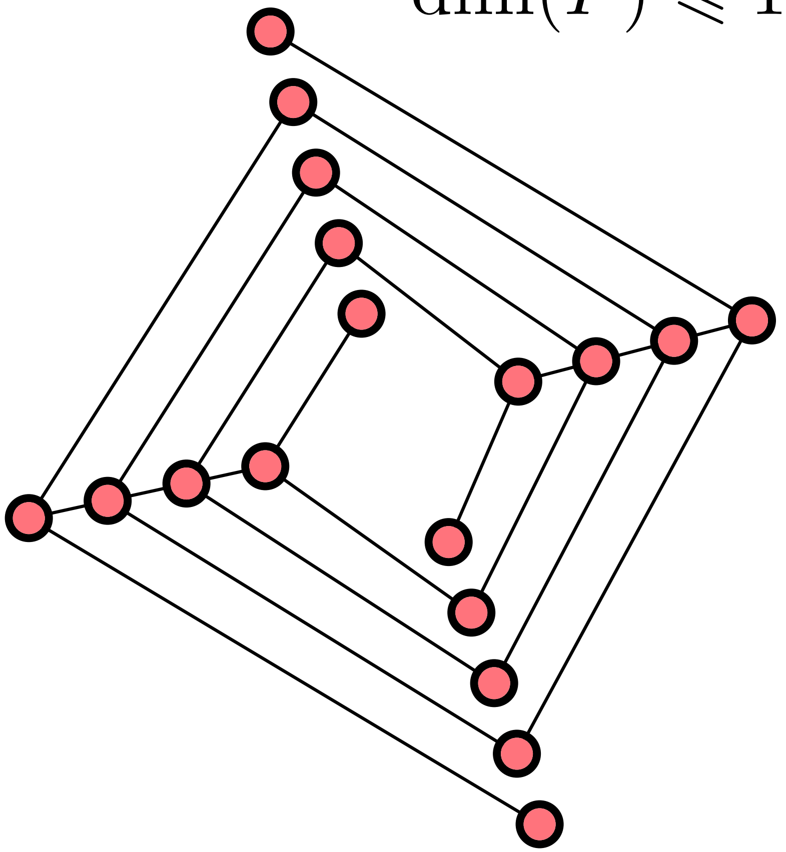

⋆ is $\dim \in O(n^{\varepsilon})$ $\forall \varepsilon > 0$ for posets with cover graphs in a nowhere dense class? (Král')

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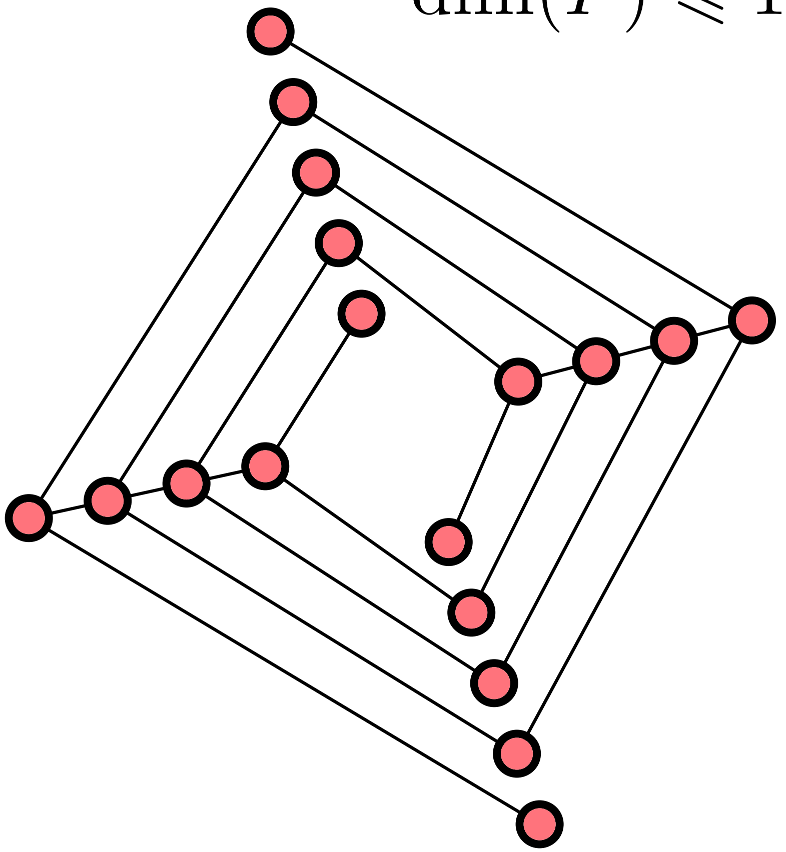

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Thank you!