On (banner, odd hole)-free graphs

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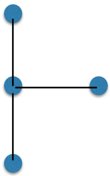
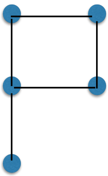
H-Free Graphs

Definition

For a graph H, a graph G is H-free, when G **does not contain** H as an **induced** subgraph.



claw



banner

Optimization on perfect graphs

Graph G is perfect if $\chi(H) = \omega(H)$ for each induced subgraph H of G.

Theorem (Lovasz)

A graph G is perfect if and only if its complement $\overline{G}$ is.

Theorem (Grotschel, Lovasz and Schrijver)

For a perfect graph G, the following parameters can be computed in polynomial time: $\chi(G), \omega(G), \alpha(G), \theta(G)$.

χ: chromatic number
ω: clique number
α: stability number
θ: clique cover number
For a graph G, $\chi(G) = \theta(\overline{G})$, and $\omega(G) = \alpha(\overline{G})$.

Perfect graphs

Theorem (Chudnovsky, Robertson, Seymour, and Thomas 2006)

A graph is perfect if and only if it contains no odd holes or odd anti-holes.

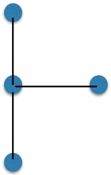
Odd holes: holes with odd length, a hole is an induced cycle with at least four vertices

Related Graph Classes and Recognition

Perfect: (odd-hole,odd-anti-hole)-free: **polytime** [Chudnovsky, Cornuejols, Liu, Seymour, Vušković; Combinatorica 2005]
Odd-hole-free; i.e., $(C_5, C_7, \ldots)$-free: **OPEN**

Banners and Odd Holes

Chvátal and Sbihi (1988) proved a structural theorem on claw-free perfect graphs.
We will deal with (banner, odd hole)-free graphs



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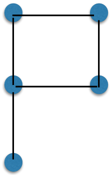


banner

Combinatorial Optimization Problems

–	Clique	Stable Set	Colouring	Clique Cover
perfect	P	P	P	P
Odd hole-free	NP-hard	?	NP-hard	NP-hard
Banner-free	NP-hard	NP-hard	NP-hard	NP-hard
(Banner, odd hole)-free	NP-hard	P	P	NP-hard

From our structural result, we design a poly-time recognition theorem for (banner, odd hole)-free graphs.

Structure of claw-free perfect graphs

Lemma (Ben Rebea's Lemma)

Let G be a connected claw-free graph. If G contains an odd anti-hole, then G contains a C_5.

Theorem (Chvátal & Sbihi)

Let G be a (claw, odd hole, odd anti-hole)-free graph. Then, one of the following holds

1. *G has a clique cutset.*
2. *$\alpha(G) \geq 3$ and G contains no hole of length at least five.*
3. *G is elementary*

Elementary graphs can be recognized in polynomial time.
This gives a recognition algorithm for claw-free perfect graphs.

Homogeneous set decomposition

A set H of vertices of a graph G is homogeneous if

H has at least two vertices but not all vertices of G, and

every vertex in $G - H$ is adjacent to all vertices of H or no vertex of H.

Given H of G, decompose G into

$G_1 = H$

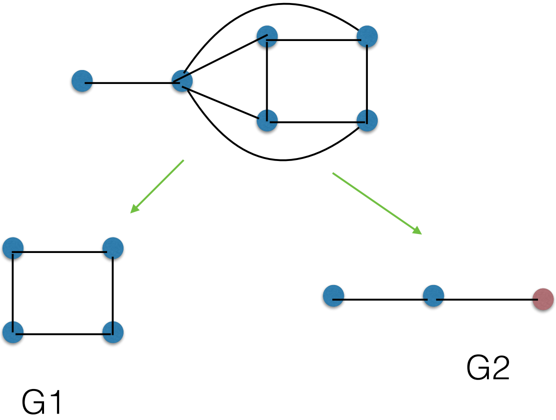
$G_2 = G - (H - h)$ for a vertex $h \in H$.

G is odd hole-free if and only if both G_1 and G_2 are.

Homogeneous sets are also called modules.

The theory of modular decomposition shows that the number of graphs produced by the decomposition is linear.

homogeneous set decomposition



Our Results

Theorem

For a (banner, odd hole)-free graph G, one of the following hold:

1. *G is a perfect.*
2. $\alpha(G) \leq 2$.
3. *G contains a homogeneous set.*

Using this theorem we design polynomial-time algorithms for

- recognizing (banner, odd hole)-free graphs.
- find a minimum coloring of a (banner, odd hole)-free graph.
- find a maximum stable set of a (banner, odd hole)-free graph.

Our Results

Theorem

For a (banner, odd hole)-free graph G, one of the following hold:

1. *G is a perfect.*
2. $\alpha(G) \leq 2$.
3. *Every odd anti-hole of G is contained in a homogeneous set H with* $\alpha(G) \leq 2$.

Proof

Let G be a (banner, odd hole)-free graph.
Assume G is not prefect, and $\alpha(G) \geq 3$.
G has an odd anti-hole A.

- No two vertices of A extends into a co-triangle (stable set on three vertices).
- Some two vertices of A extends into a co-triangle.

In each case, we will show A belongs to a homogeneous set.

Proof

Lemma

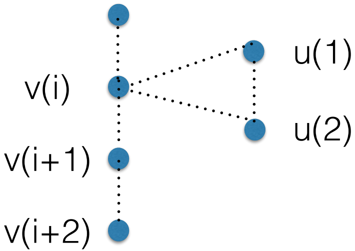
Let G be a (banner, C_5)-free graph containing an odd anti-hole A such that

- $\alpha(G) \geq 3$, *and*
- *no co-triangle of G contains two vertices of A.*

Then the complement $\overline{G}$ of G contains a triangle-free component O that contains all of A.

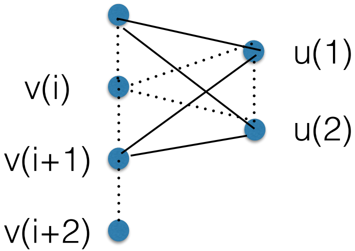
Proof

Case: no two vertices of A extends into a co-triangle
We will prove no vertex of A belongs to a co-triangle
Proof by contradiction



Proof

Case: no two vertices of A extends into a co-triangle

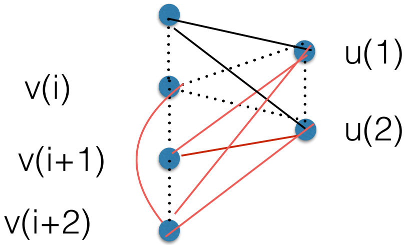


v(i+1) sees u(1), u(2), by assumption

Proof

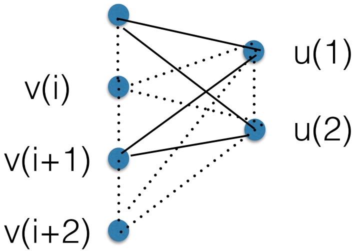
Case: no two vertices of A extends into a co-triangle

if v(i+2) sees u(1), u(2),
then banner



Proof

Case: no two vertices of A extends into a co-triangle

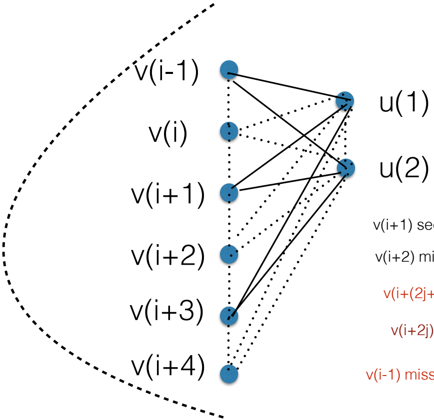


v(i+1) sees u(1), u(2), by assumption

v(i+2) misses u(1), u(2), else banner

Proof

Case: no two vertices of A extends into a co-triangle



v(i+1) sees u(1), u(2), by assumption

v(i+2) misses u(1), u(2), else banner

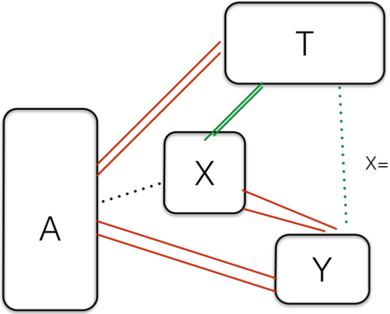
v(i+(2j+1)) sees u(1), u(2), for all j

v(i+2j) misses u(1), u(2), for all j

v(i-1) misses u(1), u(2), a contradiction

Proof

Know: vertices of A do not belongs to co-triangle



T= all co-triangles

X= vertices in G-(A U T) that is T-universal

Y= vertices in G-T that is not T-universal

Proof:
All edges A—T
All edges A—Y
All edges X—Y

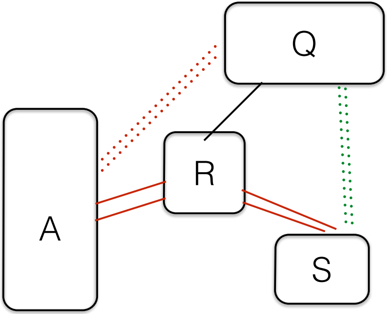
The complement of G is disconnected
A belong to a component C with no co-triangle
and C is a homogeneous set

second lemma

Lemma

Let G be a (banner, C_5)-free graph with an odd antihole A. Suppose some two vertices of A belong to a co-triangle. Then there is a homogeneous set H of G such that H contains A and no co-triangle of $G[H]$ contains two vertices of A.

Proof



Q=vertices v forming a co-triangle with two vertices of A

R = vertices in G-(A+Q+S)
R has some neighbours in Q

S= vertices in G-(A+Q) that misses all of Q

Proof:
No edges A—Q
All edges R—A
All edges R—S

H= A+S is homogeneous
No two vertices of A belong to a co-triangle of H
(Can now apply first lemma to H)

recognizing (banner, odd hole)-free graphs

Let G be an input graph.

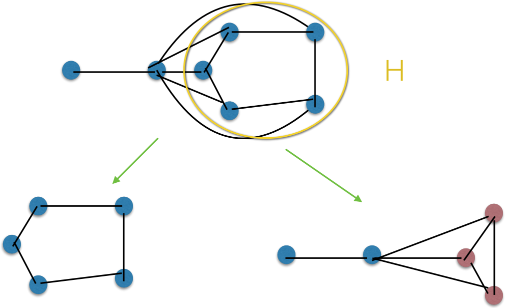
- check that G is banner-free.
- check that $\alpha(G) \leq 2$
  - if YES, look for a C_5.
  - if NO, continue
- check that G is perfect
  - if YES, then G is (banner, odd hole)-free.
  - if NO, find an odd anti-hole A
- find a homogeneous set H containing A
- decompose G into two graphs G_1, G_2.
- recursively check that G_1 and G_2 are (banner, odd hole)-free
- G is (banner, odd hole)-free if and only if both G_1, G_2 are.

Modular decomposition

In $O(n + m)$-time, a graph can be recursively decomposed by homogeneous set into induced prime graphs.
Our algorithm can be made more efficient by

- Applying the modular decomposition to obtain $O(n)$ prime graphs.
- Check that each prime graph is perfect or has $\alpha \leq 2$.

coloring (banner, odd hole)-free



Color G1 first
X(G1) = 3

in G2, substitute H by K3
Recursively color G2

χ-bounded graphs

- A graph G is 2-divisible if the vertex-set of each induced subgraph H of G with at least one edge can be partitioned into two sets V_1, V_2 such that $\omega(V_i) \leq \omega(H)$.
- 2-divisible graphs G have $\chi(G) \leq 2^{\omega(G)-1}$.
- odd-hole-free graphs are not 2-divisible

χ-bounded graphs

Conjecture (C. McDiarmid, C.T.H.)

A graph is 2-divisible if and only if it is odd-hole-free.

Conjecture (C. McDiarmid, C.T.H.)

Odd-hole-free graphs G have $\chi(G) \leq 2^{\omega(G)}$.

Theorem (Scott, Seymour)

Odd-hole-free graphs G have $\chi(G) \leq 2^{2^{\omega(G)}}$.

2-divisibillity

Theorem

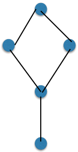
(banner, odd hole)-free graphs are 2-divisible.

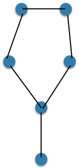
This implies $\chi(G) \leq 2^{\omega(G)-1}$.
Nicolas Trotignon remarked $\chi(G) \leq \omega(G)^2$.

pan-free graphs

- A k-pan is a C_k and a vertex with one neighbour in the C_k, $k \geq 4$.
- The banner is a 4-pan.
- A pan is a k-pan for some k.

Brandstadt, Lozin, and Mosca designed polynomial-time algorithm for finding a maximum stable set in a pan-free graph.





(pan, even hole)-free graphs

Theorem

For a (pan,even-hole)-free graph G, one of the following hold:

1. *G is a clique.*
2. *G contains a clique cutset.*
3. *G is a unit circular arc graph*
4. *G is the join of a clique and a unit circular arc graph.*

- Recognition in $\mathcal{O}(nm)$ time.
- Colouring in $\mathcal{O}(n^{2.5} + nm)$ time.

Open problems

Open Problems:

- Odd-hole-free: recognition, maximum stable set, structural characterization.
- Even-hole-free: maximum stable set, colouring, minimum clique cover.
- (claw,even-hole)-free: clique cover.

THANK YOU!