Fixed Parameter Algorithms for Completion Problems on Plane Graphs

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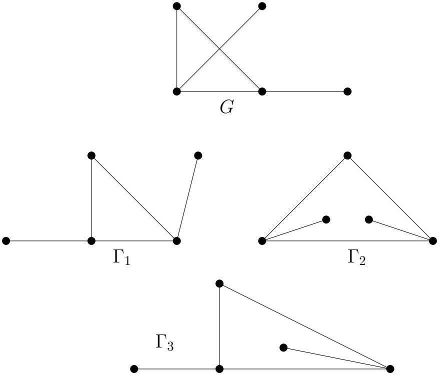
Completion Problems

Problem: Π
Input: Graphs $G_1, \ldots, G_l$
Question: Do the graphs have a specified property P ?

Problem: Π-Completion
Input: Graphs $G_1, \ldots, G_l$
Question: Can we add some edges to one or more of the graphs so that they will have the property P ?

Planar and Plane Graphs

For example:



What is FPT?

A parameterized problem is **Fixed-Parameter Tractable** (FPT) if there is an algorithm that solves it in polynomial time with respect to the size of the problem, disregarding the effect of the parameter.

Problem:	Π
Input Size:	n
Parameter:	k
FPT Algorithm:	$f(k) \cdot n^{O(1)}$

The Subgraph & Minor Isomorphism Problems

Containment relation:	$\leq$
Input:	H and G
Question:	Is $H \leq G$?

NP-complete

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	General	Planar
S.I.	W[1]-hard	$2^{O(k)} \cdot n$ (Eppstein 1999)
M.I.	$g(k) \cdot n^3$ (Robertson & Seymour 1995)	$O(2^{O(k)} \cdot n + n^2 \cdot \log n)$ (Adler et al. 2010)

where $n = \lvert V(G) \rvert$ and $k = \lvert V(H) \rvert$.

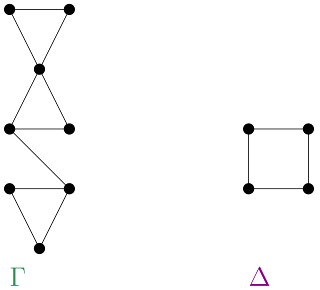
The Plane Subgraph Completion Problem

PLANE SUBGRAPH COMPLETION (PSC)

Input: A "host" plane graph Γ and a "pattern" **connected** plane graph Δ.

Parameter: $k = |V(\Delta)|$

Question: Can we add edges to Γ so that it contains a subgraph topologically isomorphic to Δ **while preserving the embedding**?



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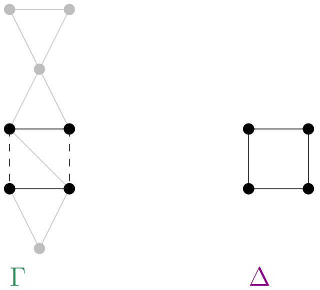

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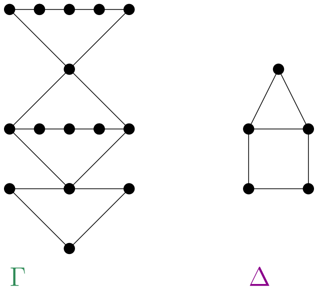

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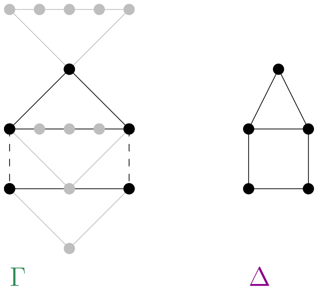

Our Results

If $k := |V(\Delta)|$ and $n := |V(\Gamma)|$, we give:

- an FPT algorithm for PSC that runs in time $2^{O(k \log k)} \cdot n^2$ and
- an FPT algorithm for PTMC that runs in time $g(k) \cdot n^2$.

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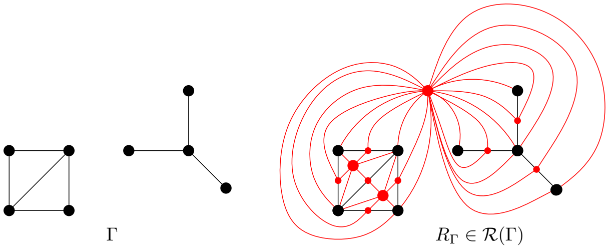
Remark. In fact we can even solve more general problems: we can ask that the pattern graph Δ be given as a **planar** graph and check whether **any** of its embeddings can be found in the host.

Subdivided Radial Enhancement

Γ *subdivided radial enhancement* R_Γ



Some Observations

Let's consider some facts about this construction.

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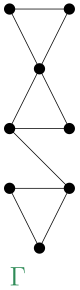
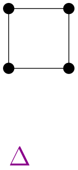
Whitney's Theorem (1932): Any 3-connected planar graph admits a unique embedding on the plane (up to topological isomorphism).

The PSC-Algorithm

Input:

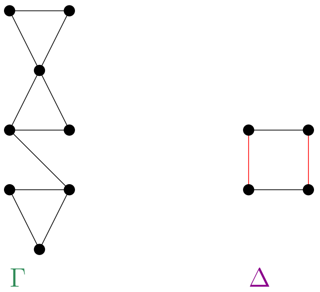




The PSC-Algorithm

Step 1: Guess which edges of Δ (red) are missing from Γ.

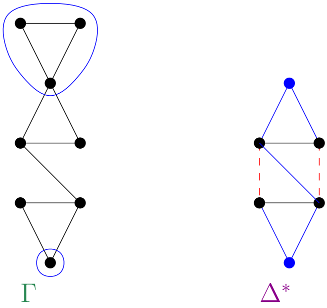
$O(2^k)$ time



The PSC-Algorithm

Step 2: Guess a supergraph Δ^* of Δ with extra (blue) vertices and edges in some faces that represent vertices and edges of Γ inside the corresponding faces. Then remove the red edges.

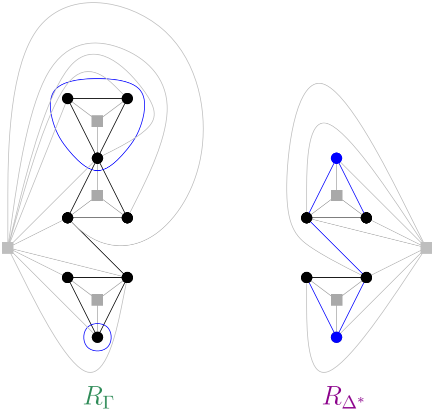
$O(2^{k \log k})$ time



The PSC-Algorithm

Step 3: Consider R_{Γ} arbitrarily and "guess" an R_{Δ^*}.

$O(n + 2^k)$ time



The PSC-Algorithm

Step 4: **Enhance** R_{Γ} and R_{Δ^*} twice. Then $Q(\Gamma)$ and $Q(\Delta)$ are **3-connected** and **uniquely embeddable**.

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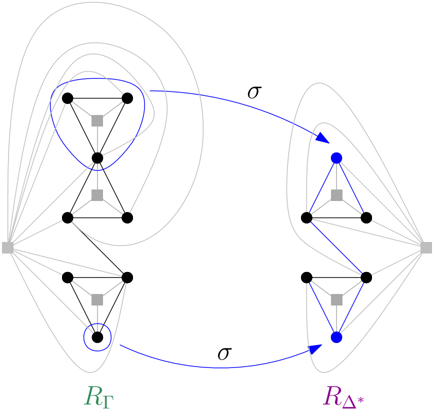
Step 5: Pick a vertex u of Γ and contract everything in $Q(\Gamma)$ that is at a distance greater than $\mathbf{diam}(Q(\Delta)) = O(k)$ from u. It is easy to prove that the resulting graph $Q_u(\Gamma)$ has treewidth $\leq 3 \cdot \mathbf{diam}(Q(\Delta)) = O(k)$.

$O(n)$ time

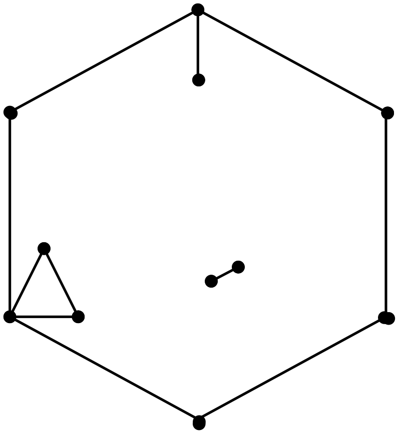
The PSC-Algorithm

Step 6: Use a modified algorithm by Adler et al. (2011) to check whether **the planar graph** $Q_u(\Gamma)$ contains **the planar graph** $Q(\Delta)$ as a *special contraction*.

$\leq n$ steps



Cylindrical Enhancement



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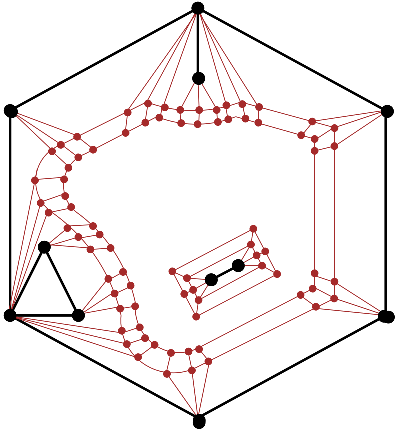

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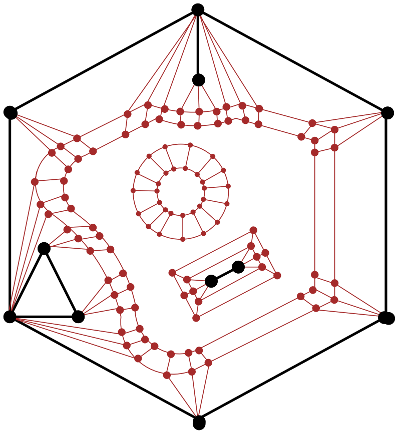

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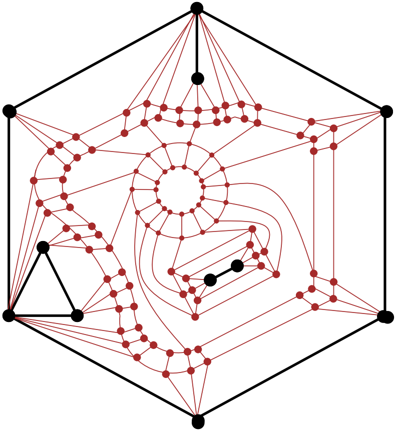

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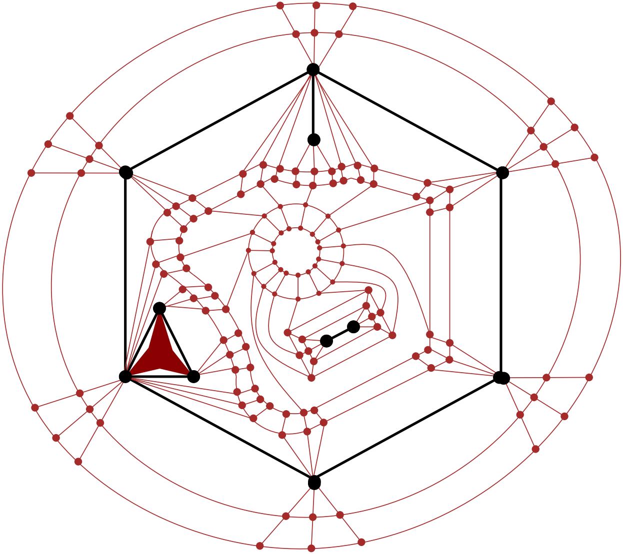
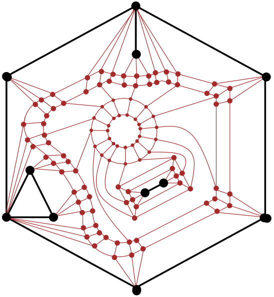


Cylindrical Enhancement

The resulting graph Γ^c has $O(n)$ vertices.



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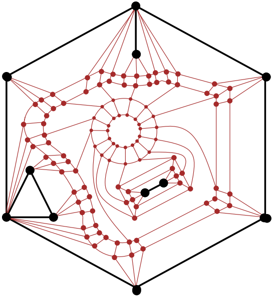

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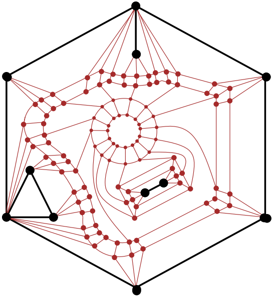

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- This relation can be expressed in MSOL.

Rooted Disjoint Paths

- To prove the previous claim, we use a result by Adler et al. (2011) which states that the number of edges that need to be added in each face in order to find k disjoint paths is bounded by $f(k)$.

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- Using this result, we can solve the PLANAR ROOTED TOPOLOGICAL MINOR COMPLETION PROBLEM even for disconnected patterns and therefore the PLANAR DISJOINT PATHS COMPLETION PROBLEM.

Bounding the Tree-width: The Irrelevant-Edge Algorithm

We combine two known algorithms in order to find an irrelevant edge in the graph (i.e., an edge whose removal results in an equivalent instance) in time $g(k) \cdot n$:

- by Golovach,Kamiński,Maniatis,Thilikos (2015), we find a large wall with some special properties in the graph and
- by Kaminski,Thilikos (2012), we find an irrelevant edge in the wall.

The PTMC-Algorithm

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Step 2: If $\mathbf{tw}(\Gamma^c) \leq f(k)$, proceed to step 3. Otherwise, find an irrelevant edge in Γ^c and remove it. Repeat until $\mathbf{tw}(\Gamma^c) \leq f(k)$.

$\leq g(k) \cdot n^2$ time

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Step 3: **Enhance** twice $\Gamma^{c\,-}$ and Δ, resulting in $Q(\Gamma)$ and $Q(\Delta)$.

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Step 4: Use Courcelle's algorithm to check whether $Q(\Delta) \leq^* Q(\Gamma)$.

$\leq h(k) \cdot n$ time

Side-Results / Future work

- We can modify the PSC-algorithm to check if the pattern graph appears as **induced subgraph** in the host.
- Although the PTMC-algorithm works for **minors** as is, we can modify it slightly to obtain a linear algorithm (w.r.t. n).
- Try to drop the super-exponential factor $2^{O(k \log k)}$ of PSC to just exponential. A better way to “guess" the blue parts in the pattern will be needed.

Thank you for your attention!