The Parameterized Complexity of Finding Paths with Shared Edges

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VIP-Routing

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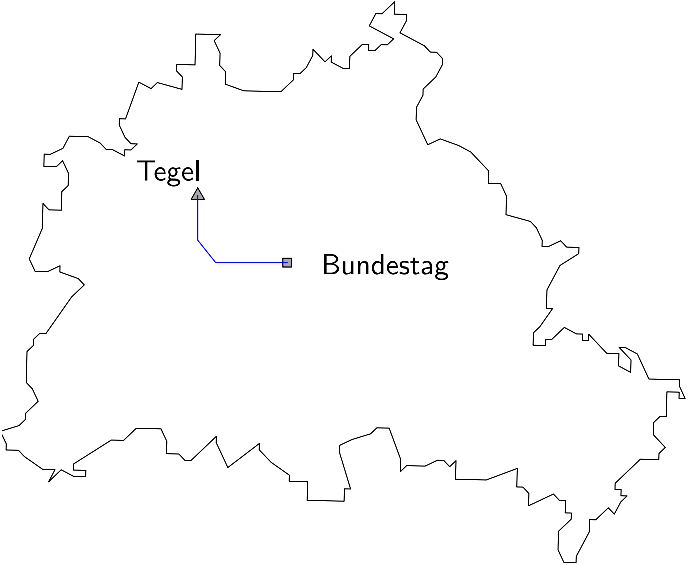

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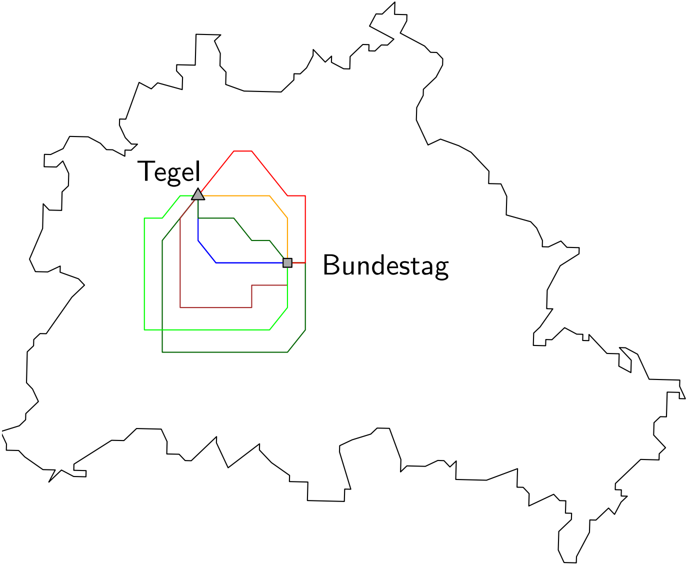

The Problem

Problem: MINIMUM SHARED EDGES (MSE)

Input: A simple, undirected graph $G = (V, E)$, $s, t \in V$, and two integers $p \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Question: Are there p s-t paths in G that share at most k edges?

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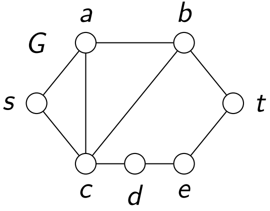

Are there $p = 3$ s-t paths in G that share at most $k = 2$ edges ?

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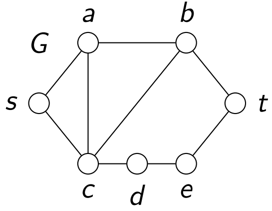

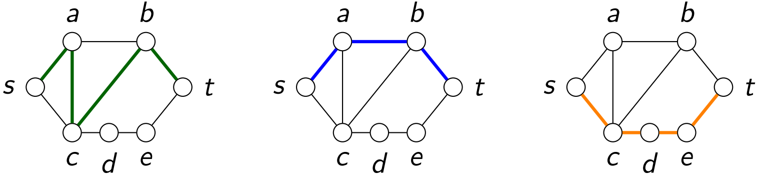

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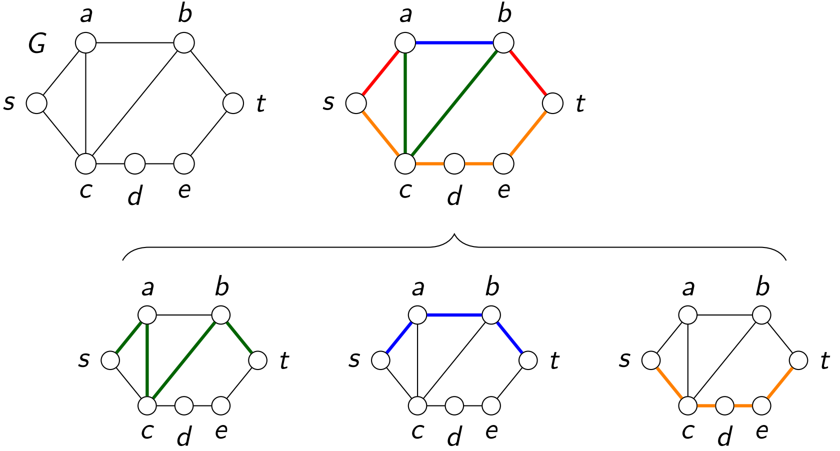

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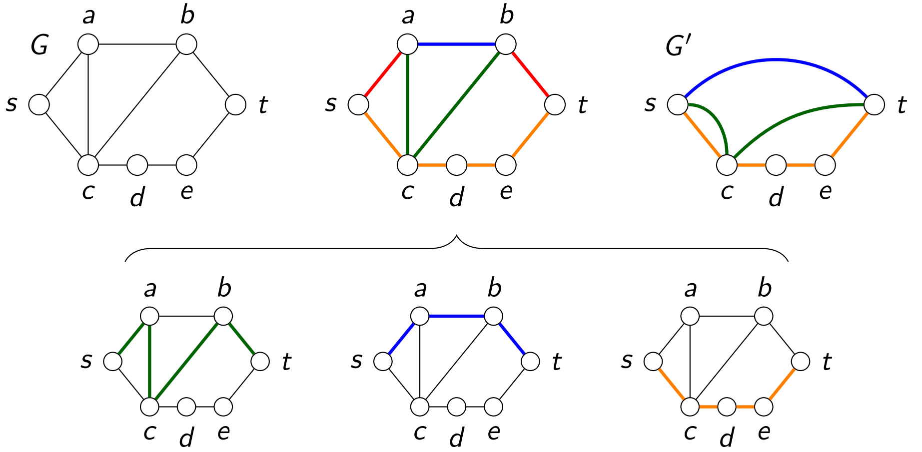

Related Work

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Aoki et al. [COCOA '14]	MINIMUM VULNERABILITY on undirected graphs	solvable in $p^{f(\text{tw})} \cdot n^{O(1)}$ time; $\text{MV}(p)$ is FPT on chordal graphs

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MINIMUM SHARED EDGES *is NP-complete, even on graphs of maximum degree five.*

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Minimum Shared Edges *is NP-complete, even on graphs of maximum degree five.*

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$\mathrm{MSE}(p)$ is FPT.

Theorem

MINIMUM SHARED EDGES *is fixed-parameter tractable with respect to the number p of paths.*

Strategy of Proving "MSE(p) is FPT"

Instance: (G, s, t, p, k)

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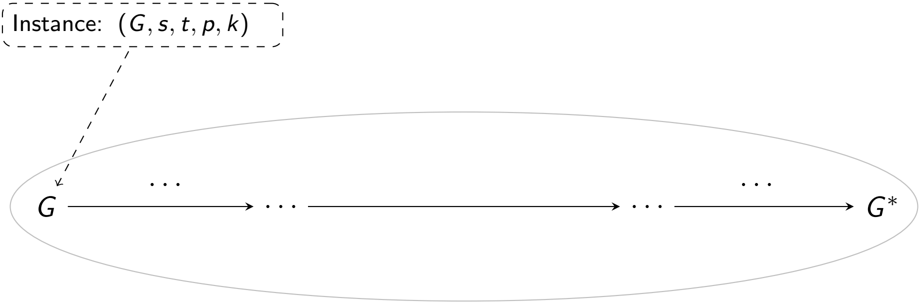

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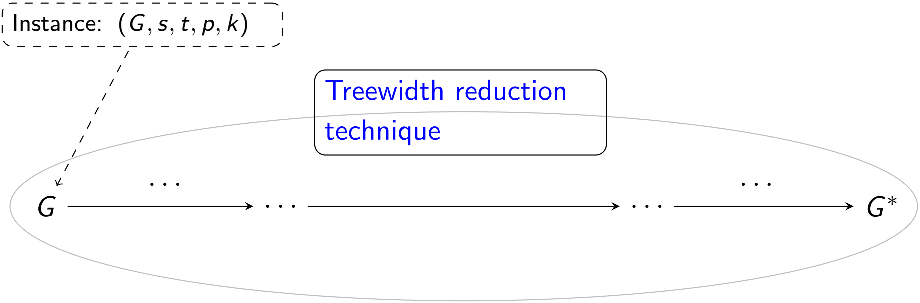

The Treewidth Reduction Technique

Theorem (Marx et al. [TALG '13, Theorem 2.15])

Let G be a graph, $T \subseteq V(G)$, and let ℓ be an integer. Let C be the set of all vertices of G participating in a minimal s-t separator of size at most ℓ for some $s, t \in T$.

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(2) *For every $s, t \in T$, a set $L \subseteq V(G^*)$ with $|L| \leq \ell$ is a minimal s-t separator of G^* if and only if $L \subseteq C \cup T$ and L is a minimal s-t separator of G.*

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(4) *$G^*[C \cup T]$ is isomorphic to $G[C \cup T]$.*

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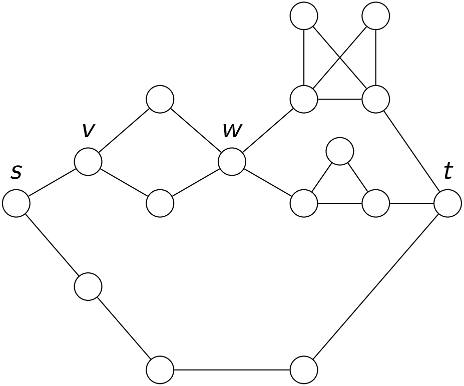

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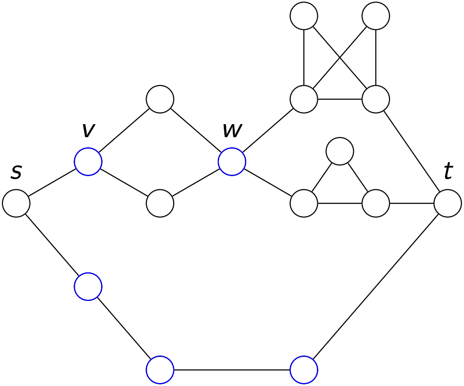

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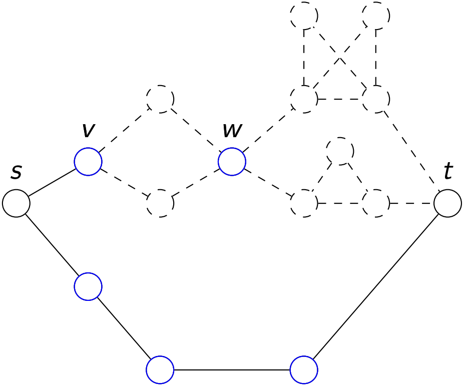

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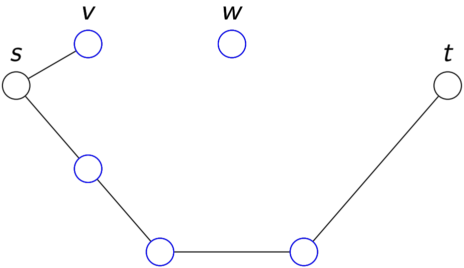

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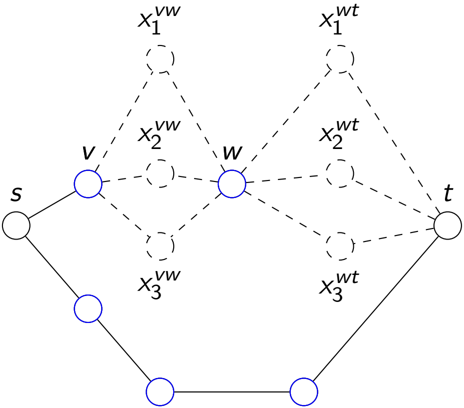

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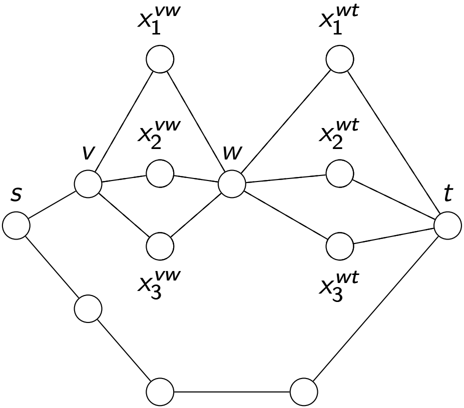

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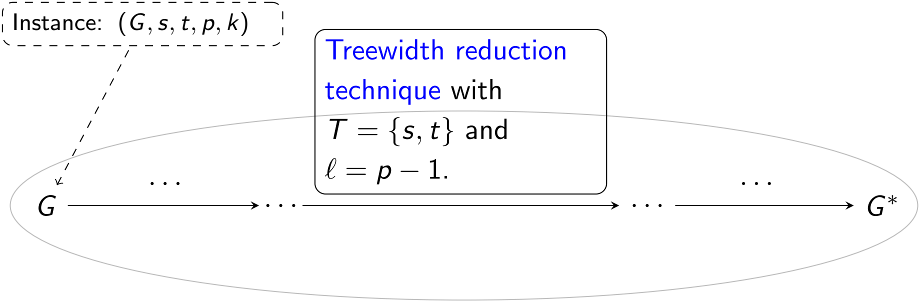

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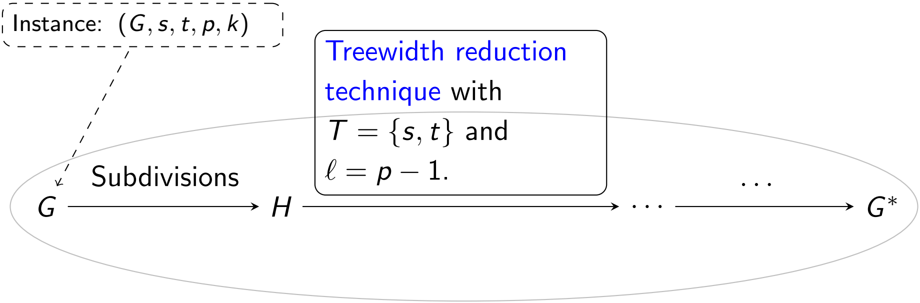

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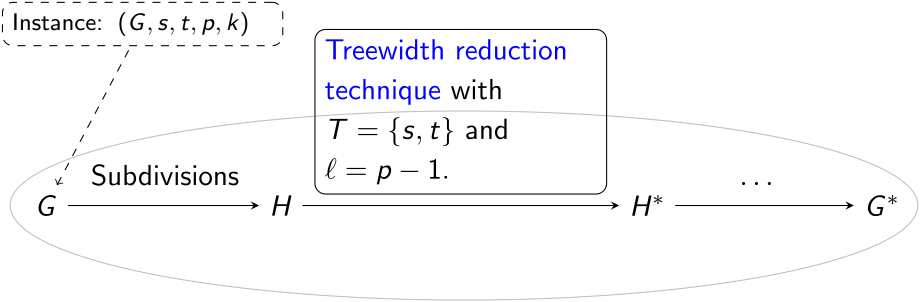

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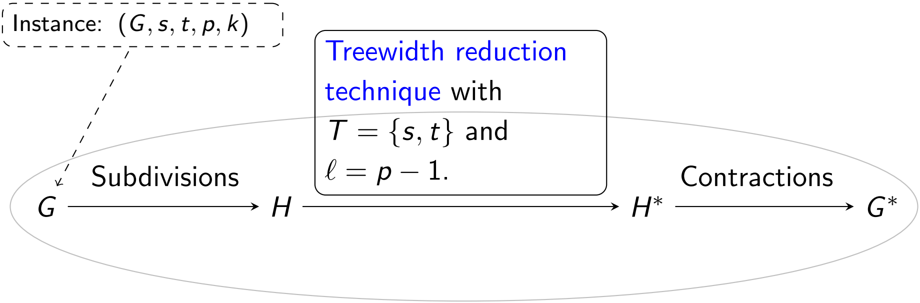
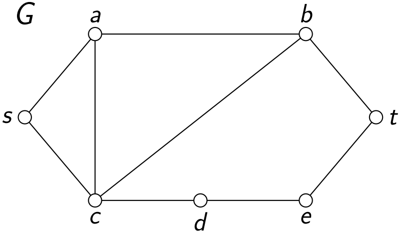


From G to G^* (with $\ell = p - 1 = 2$)



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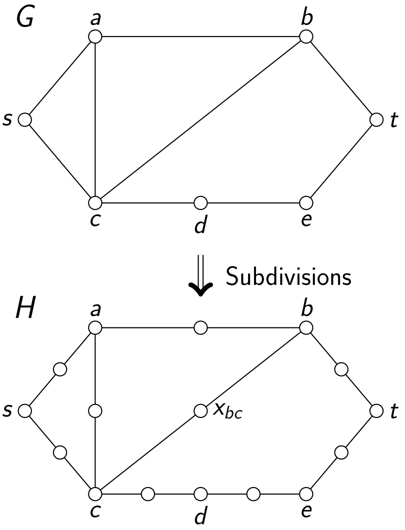

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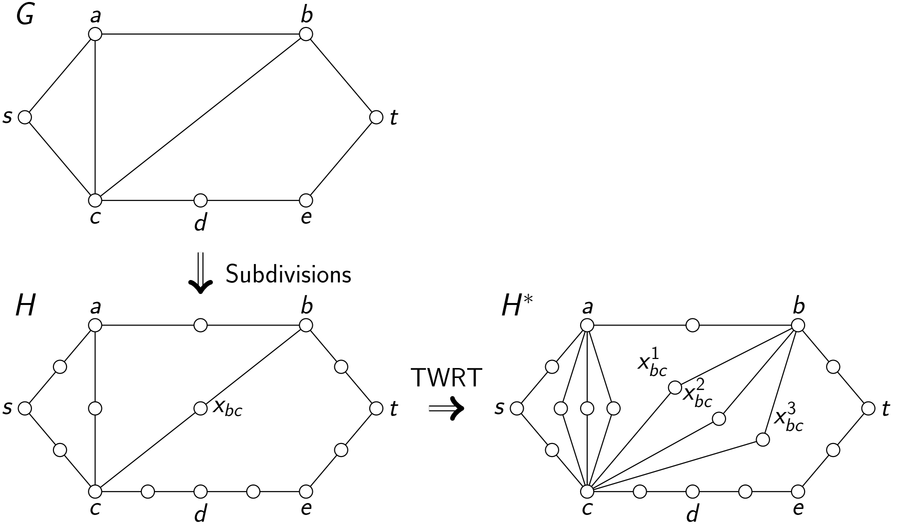

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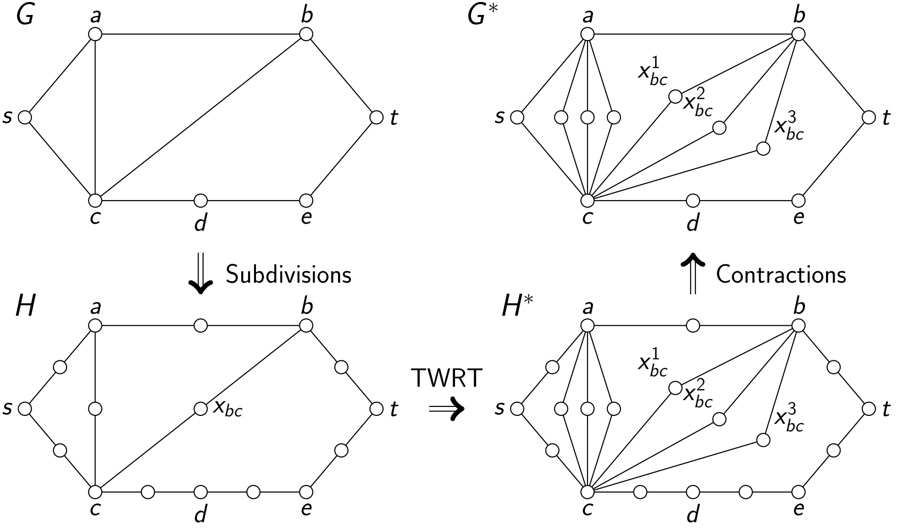

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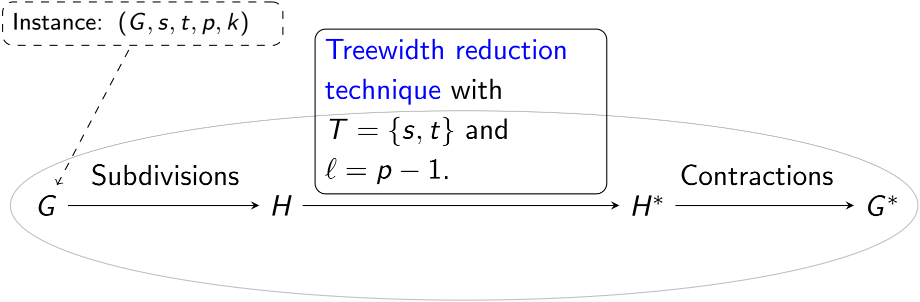

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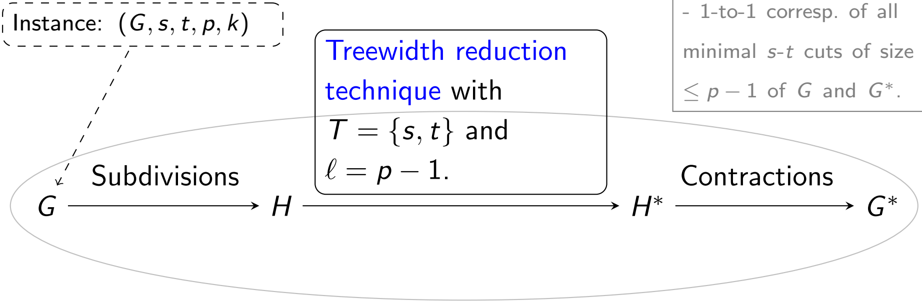

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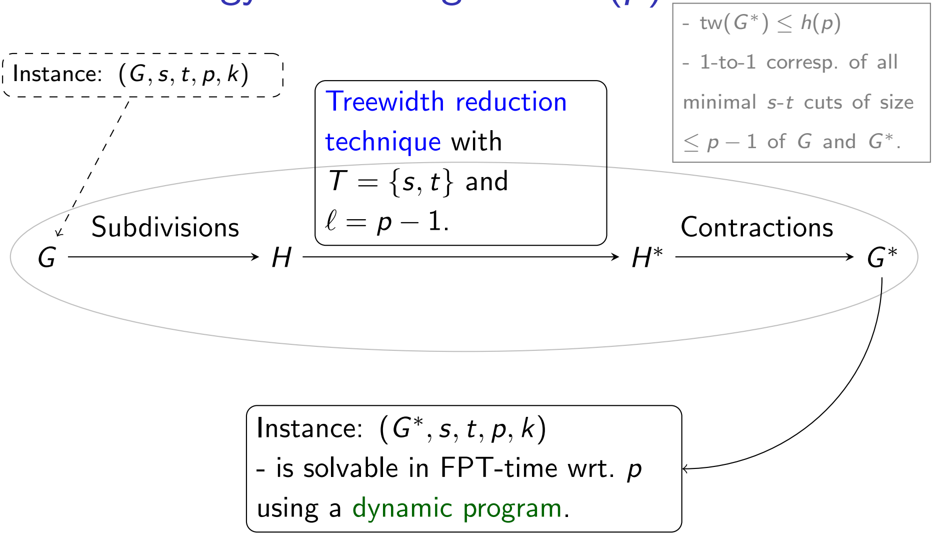

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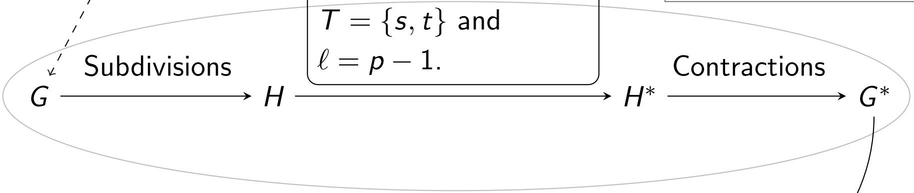
Treewidth reduction technique with $T = \{s, t\}$ and $\ell = p - 1$.



Instance: (G^*, s, t, p, k)
- is solvable in FPT-time wrt. p using a dynamic program.
- is yes-instance if and only if (G, s, t, p, k) is yes-instance.

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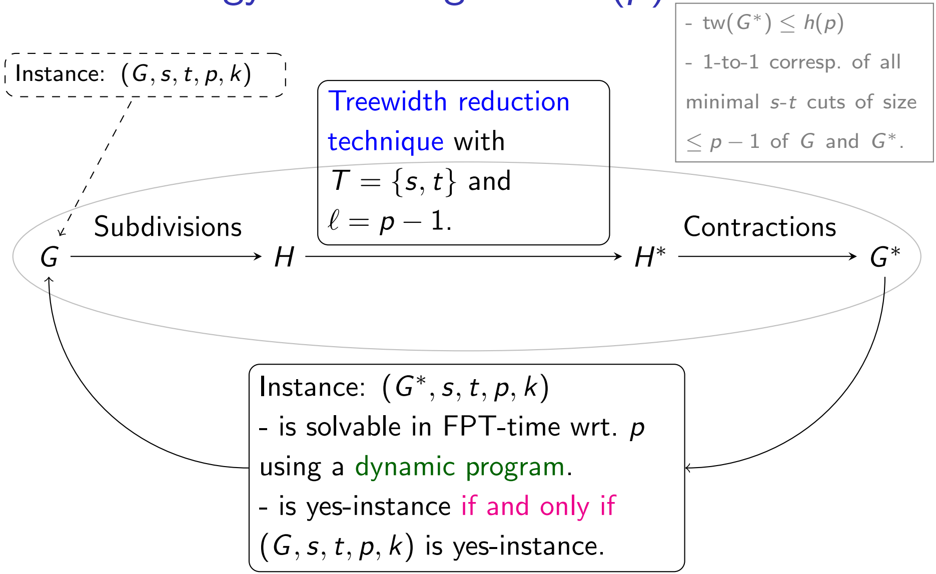

Introduction

The Number of Paths

Conclusion and Remarks

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- Further research: complexity of MSE on *special* planar graphs, e.g. grids with holes.

Thank you.

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