Intersection Graphs of Non-crossing Paths

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GROW 2015, Modane, France.

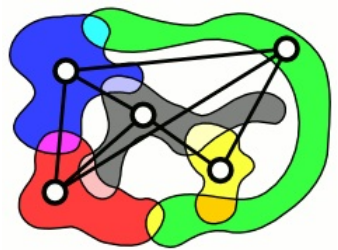
Outline

Outline

Set Representations of Graphs

Definition

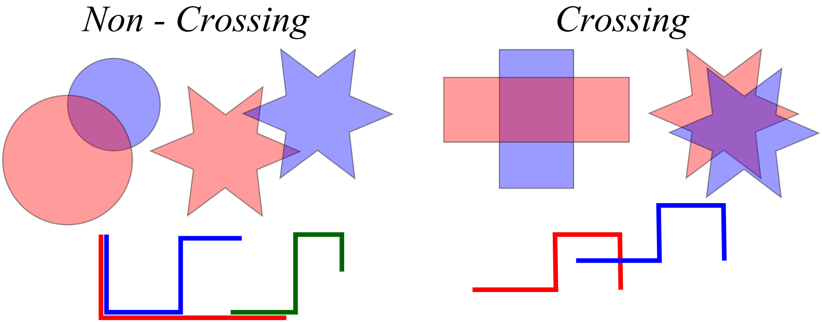
For a collection $\mathbb{S}$ of sets $S_1, ..., S_n$, the ***intersection graph*** $G(\mathbb{S})$ of $\mathbb{S}$ has vertex set $\mathbb{S}$ and edge set $\{S_iS_j : i, j \in \{1, ..., n\}, i \neq j, \text{ and } S_i \cap S_j \neq \emptyset\}$.
We call $\mathbb{S}$ an ***intersection representation*** of $G(\mathbb{S})$.



http://upload.wikimedia.org/wikipedia/commons/e/e9/Intersection_graph.gif

Crossing

Two connected sets ***cross*** if their ***difference*** is disconnected.



Non-crossing Classes

Recognition

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Anything Tractable ... ?? Yes!

- **Proper Interval Graphs** $O(n + m)$ [Corneil, Kamula 1987; Deng, Hell, Huang 1996]

Classic Intersection Classes

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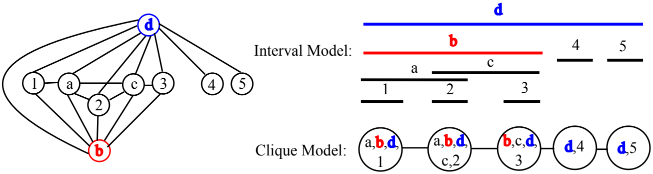
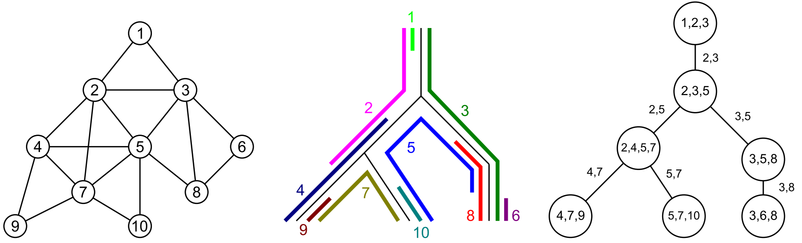


Classic Intersection Classes

- Interval Graphs: intersection graphs of subpaths of a path.
- Chordal Graphs: no induced k-cycle ($k \geq 4$) = intersection graphs of subtrees of a tree. [Gavril 1974]



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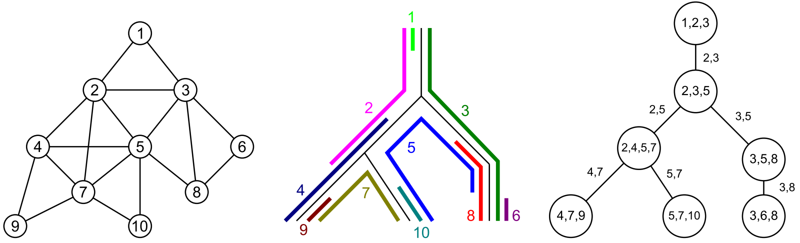

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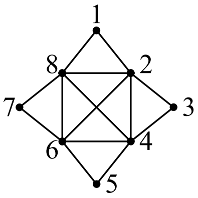

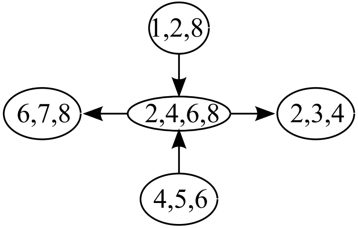

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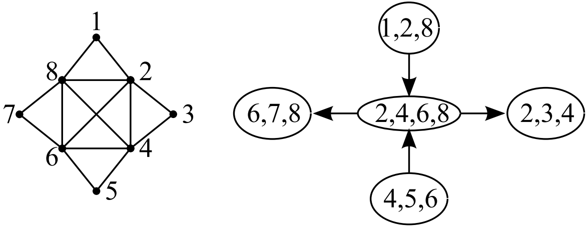
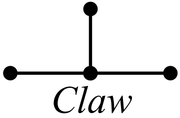
- Rooted Path-Tree Graphs: directed paths in a rooted tree.



This graph is not rootable!

Results

1. NC Path Tree = Claw-Free Chordal.



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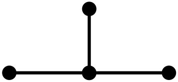
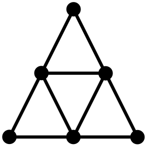


Claw



3-sun

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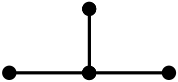

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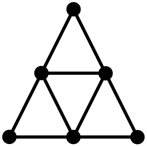

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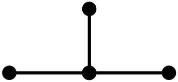
New proof of Proper interval = (Claw,3-sun,Net)-free Chordal.



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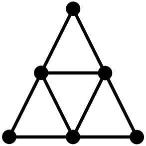

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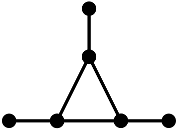

Net

Warm-up: Claw-freeness

Observation

If G is an NC-path graph then, G is claw-free.

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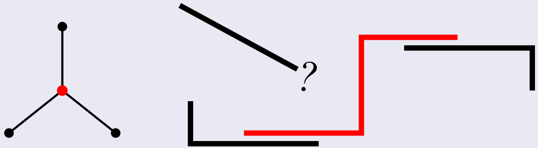
Proof.



□

Outline

Background: Intersection Graphs

Characterizations

Minimum Dominating Set

Thm: NC Path-tree = Claw-Free Chordal

NC Path-tree $\subseteq$ Claw-Free Chordal, by prev. Observation.

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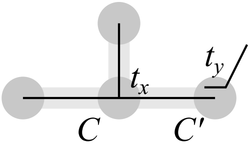

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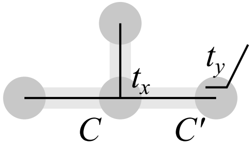
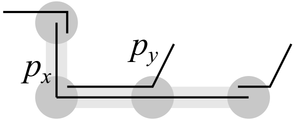
- Claim 2: These paths are non-crossing.





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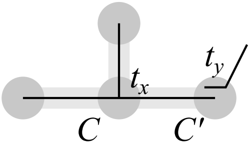
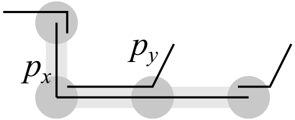


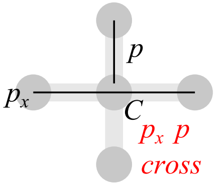


Note: this implies every clique tree of a Claw-free Graph is an NC Path-tree!

High Degree Nodes in NC Path-Tree Models

Consider a Clique Tree T, of a Claw-free Chordal graph G:

- If C is internal to a path of T, then C has degree ≤ 3.



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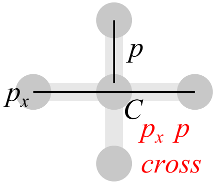

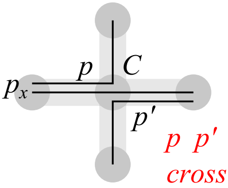

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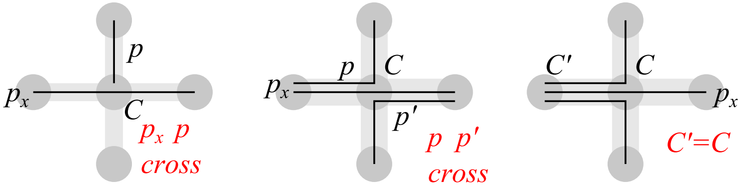
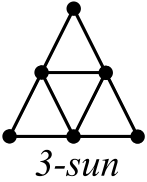
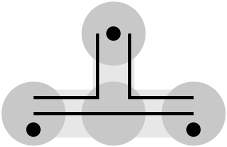


High Degree Nodes in NC Path-Tree Models

Consider a Clique Tree T, of a Claw-free Chordal graph G:

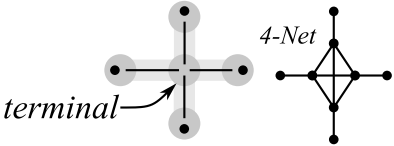
- If C is internal to a path of T, then C has degree ≤ 3.
- Moreover, C is the centre of a 3-sun.





(Claw,3-sun)-Free Chordal

Note: 3-sun is not a Directed Path-Tree Graph. Moreover, by the previous slide, forbidding the 3-sun means all degree ≥ 3 nodes are "terminals."



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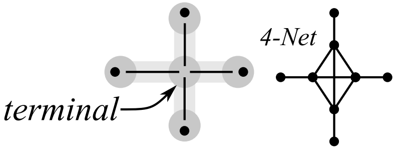

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NC Directed Path-Tree $\subseteq$ (Claw,3-sun)-free Chordal $\subseteq$ NC Rooted Path-Tree Graphs .

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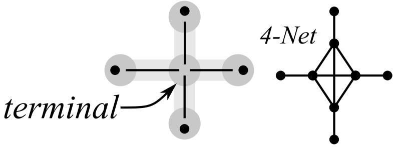

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Proof.

Simply root the representation at any "terminal". The result is an NC rooted path-tree representation. □

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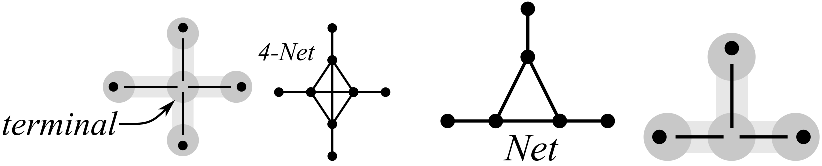

Theorem (Wenger 1967)

Proper Interval = (Claw, 3-sun, Net)-free Chordal.

Proof.

By forbidding Nets we further forbid any node in T to have degree ≥ 3. □

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Theorem

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Idea: Mark the “3-suns” in the clique tree T. Root it at a leaf. Process T “bottom-up” computing some MDSs on the parts of T between the “3-suns”.

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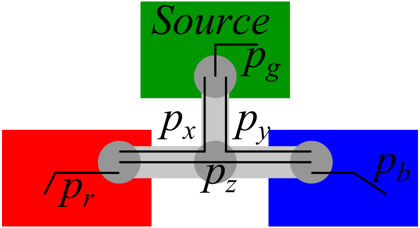

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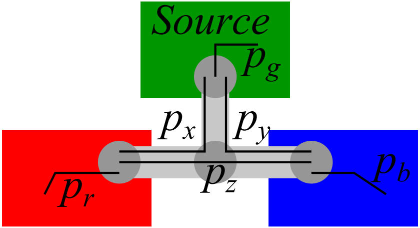

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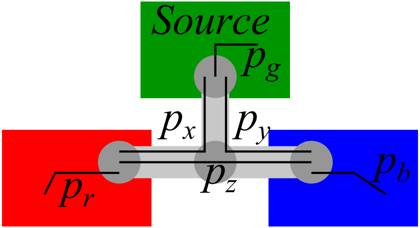

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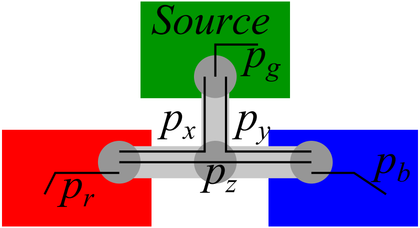
2. "cut" G on the separator xy to make the Rooted representations.



Concluding Remarks

Results (in this talk):

- NC Path-Tree = Claw-free Chordal, Directed/Rooted NC Path-Tree = (Claw,3-sun)-free Chordal.
- MDS on NC-Path-Tree in polynomial time.

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- Polynomial time recognition of NC Segment-Plane graphs.

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Open Questions:

1. What about NC Path-Grid graphs? i.e., NC-string?
2. What about NC Tree-Grid graphs?

Thank you!