Colouring graph classes with constraints on local connectivity

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Introduction

We are interesting in (proper) vertex colouring

- Let k be a fixed integer at least 3

  k-COLOURABILITY
  Input: a graph G
  Question: is there a k-colouring of G?

- k-COLOURABILITY is well known to be NP-complete for any fixed $k \geq 3$

Motivation:

- Are there (interesting) classes defined in terms of connectivity for which k-COLOURABILITY is in P?

Connectivity preliminaries

- The local connectivity $\kappa(x, y)$ of distinct vertices x and y in a graph is the maximum number of internally disjoint paths between x and y
- An xy-vertex cut is a subset Z of $V(G)$ such that x and y are in different components of $G - Z$

Theorem (Menger, 1927)

Let x and y be non-adjacent vertices. Then the minimum number of vertices in an xy-vertex cut is equal to $\kappa(x, y)$.

- A graph G is k-connected if it has at least 2 vertices and $\kappa(x, y) \geq k$ for all distinct $x, y \in V(G)$

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Brooks' Theorem

Theorem (Brooks, 1941)

For any connected graph G with maximum degree k,

- *if G is not a complete graph nor an odd cycle, then G is k-colourable*
- *otherwise, G is not k-colourable, but is $(k+1)$-colourable.*

- k-COLOURABILITY is polynomial for graphs with maximum degree k by Brooks' Theorem
- The classes we will consider contain the class of (k-connected) graphs with maximum degree at most k
- e.g. minimally k-connected graphs

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Minimally k-connected graphs

- A graph is minimally k-connected if it is k-connected, but is no longer k-connected after the removal of any edge
- We say G is k-chord-free if $\kappa(x, y) \leq k$ for all adjacent $x, y \in V(G)$

Lemma

A graph is minimally k-connected iff it is k-connected and k-chord-free.

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A hierarchy of connectivity classes

- G is *k-chord-free*
  if $\kappa(x, y) \leq k$ for adjacent $x, y \in V(G)$.
- G has *maximal local connectivity k*
  if $\kappa(x, y) \leq k$ for every $x, y \in V(G)$.
- G has *maximal local edge-connectivity k*
  if $\lambda(x, y) \leq k$ for every $x, y \in V(G)$,

where the *local edge-connectivity* $\lambda(x, y)$ of distinct vertices x and y is the maximum number of edge-disjoint paths between x and y.

Each of these classes is closed under taking subgraphs.

We can also consider the k-connected subclasses of each. In particular:

- G is *minimally k-connected* if it is k-connected and k-chord-free.

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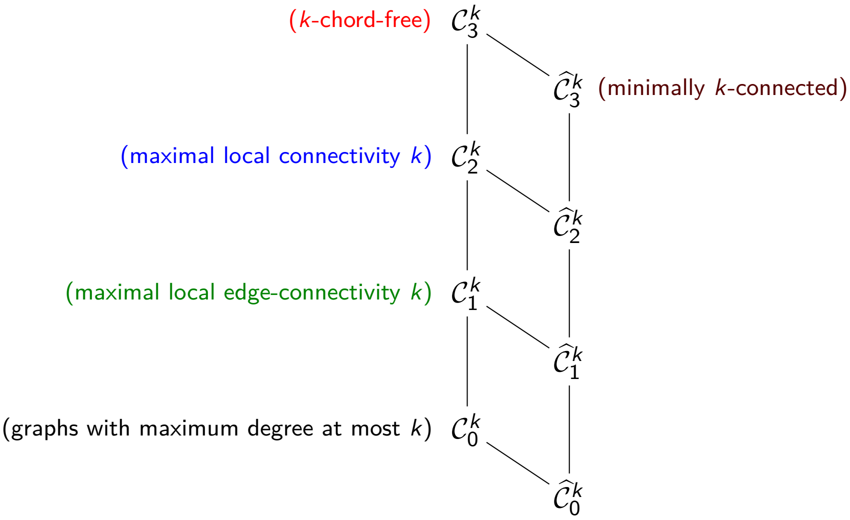
A hierarchy of connectivity classes (2)

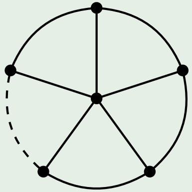


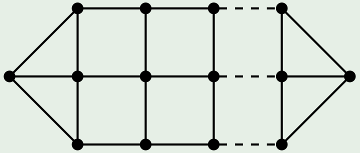
Graphs with maximal local edge-connectivity 3

What graphs are in the class?

- all cubic graphs
- all graphs with one vertex of degree more than 3 (e.g. wheels)
- some graphs with arbitrarily many vertices of degree more than 3

Example





k-chord-free graphs are $(k+1)$-colourable

Theorem (Mader, 1973)

Let G be a graph with at least one edge. Then there exists an edge $xy \in E(G)$ such that $\kappa(x, y) = \min\{d(x), d(y)\}$.

Corollary

If G is a k-chord-free graph, then G has a vertex of degree at most k.

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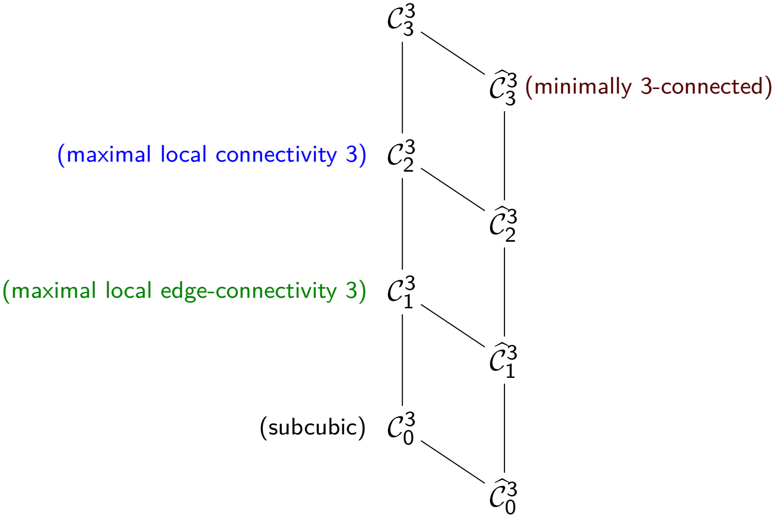
3-COLOURING complexity



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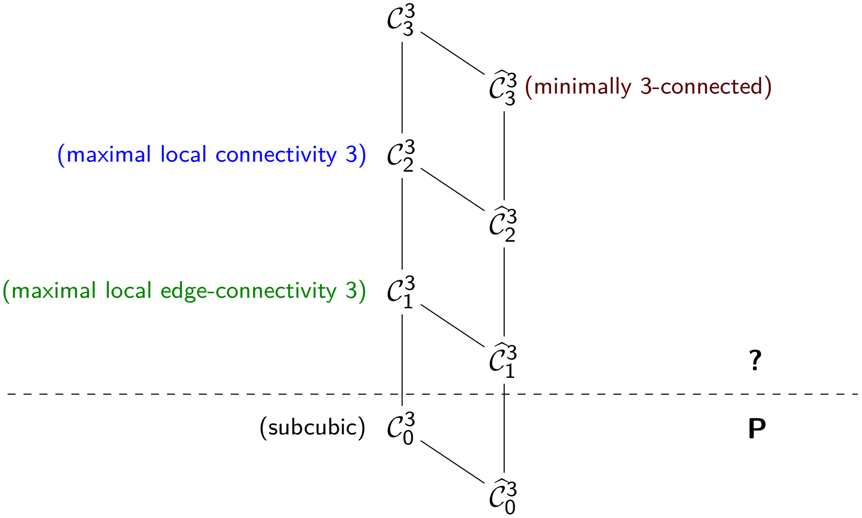

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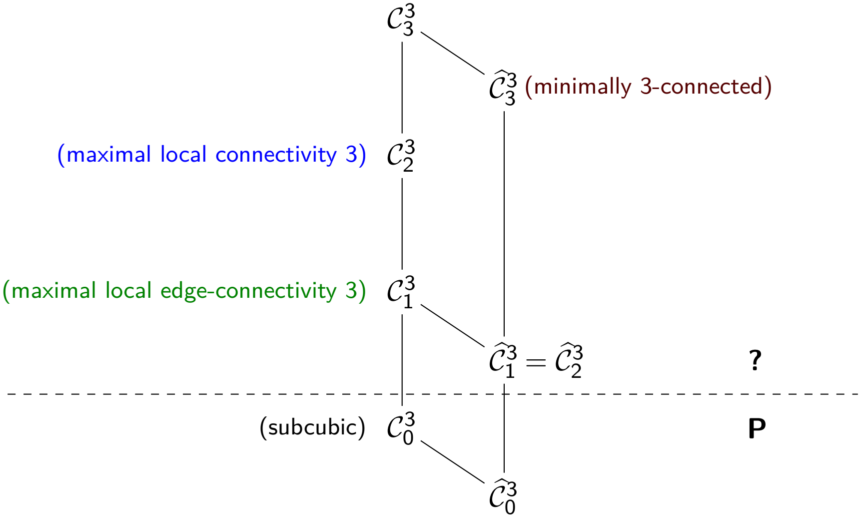
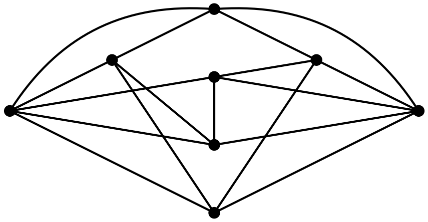


3-connected graphs with maximal local connectivity 3

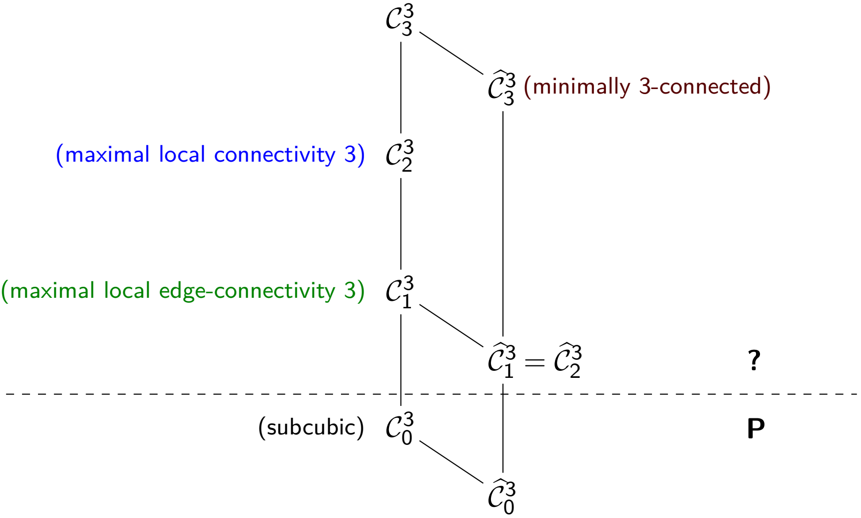
Proposition (ABHMT)

A 3-connected graph with maximal local connectivity 3 has maximal local edge-connectivity 3.

- For $k \geq 4$, a k-connected graph with maximal local connectivity k may have maximal local edge-connectivity strictly more than k



3-COLOURING complexity: results



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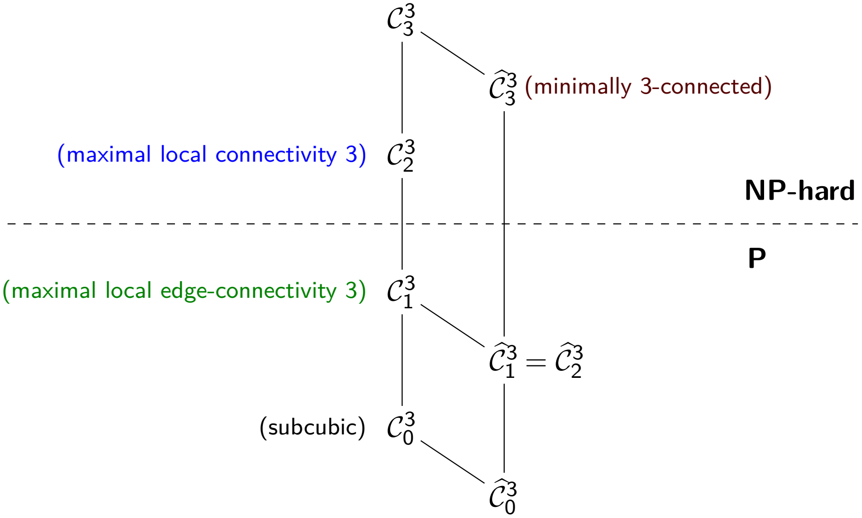
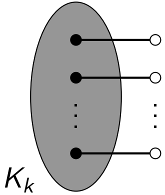


Colourability of minimally k-connected graphs

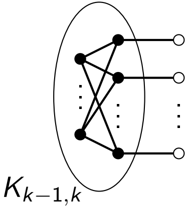
Proposition (ABHMT)

For fixed $k \geq 3$, k-COLOURABILITY remains NP-complete when restricted to minimally k-connected graphs.

- k-UNIFORM HYPERGRAPH k-COLOURABILITY is NP-complete for $k \geq 3$
- Reduce to k-COLOURABILITY on minimally k-connected graphs
- Gadgets:



(for each hyperedge)



(to ensure k-connected)

Graphs with maximal local connectivity k

Proposition (ABHMT)

For fixed $k \geq 3$, the problem of deciding if a $(k-1)$-connected graph with maximal local connectivity k is 3-colourable is NP-complete.

Proof idea for $k = 3$.

Reduce 3-COLOURABILITY (for any graph) to 3-COLOURABILITY for 2-connected graphs with maximal local connectivity 3.

Replace each high-degree vertex with a gadget like:

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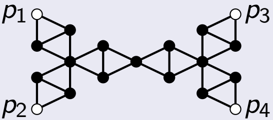
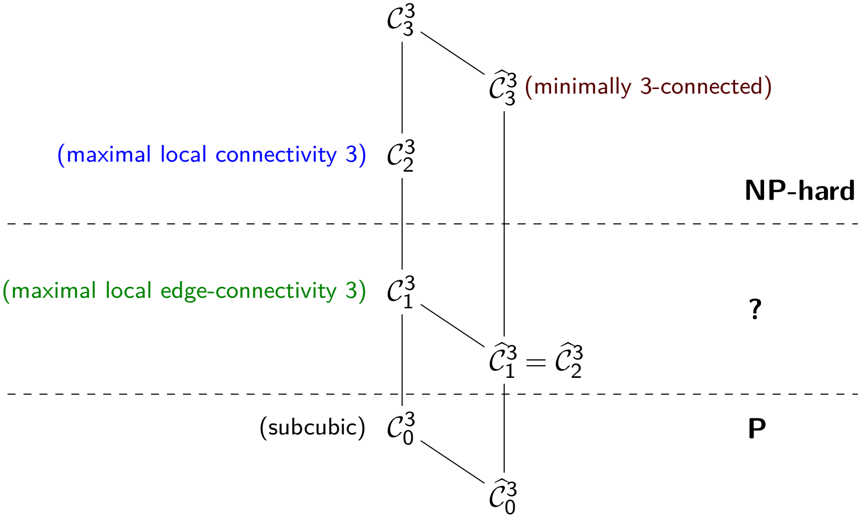


□

3-COLOURING complexity: state of play



Graphs with maximal local edge-connectivity k

Proposition (ABHMT, 2015)

k-COLOURING, restricted to k-connected graphs with maximal local edge-connectivity k, is in P.

In fact...

Theorem (ABHMT, 2015)

Let $k \geq 3$. A k-connected graph with maximal local edge-connectivity k is k-colourable if and only if it is not a complete graph nor an odd wheel.

Proof to follow.

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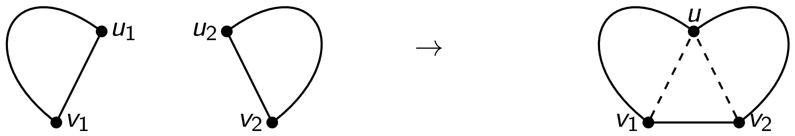
Graphs with maximal local edge-connectivity 3

When $k = 3$, we can drop the 3-connectivity requirement:

Theorem (ABHMT)

Let G be a graph with maximal local edge-connectivity 3. Then G is 3-colourable if and only if each block of G cannot be obtained from odd wheels by performing Hajós joins.

Hajós join:



Corollary

3-COLOURING *on graphs with maximal local edge-connectivity 3 is in P.*

The proof (k-connected case)

Theorem (ABHMT)

Let $k \geq 3$. A k-connected graph G with maximal local edge-connectivity k is k-colourable if and only if it is not a complete graph nor an odd wheel.

There are three ingredients to the proof:

1. If a k-connected graph has at most one vertex of degree more than k and no dominating vertices, then it is k-colourable
2. If G has more than one high-degree vertex, there is a k-edge cut S separating X from Y where X has one high-degree vertex
3. G is k-colourable if and only if there are colourings of $G[X]$ and $G[Y]$ where the colours given to the vertices incident with S are not all the same in one and all different in the other

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Ingredient 1: a variant of Brooks' theorem

Lemma

Let G be a 3-connected graph with at most one vertex of degree more than k, and no dominating vertices. Then G is k-colourable.

Lovász (1975) gave a short proof of Brooks' theorem: can easily adapt that proof in order to prove this lemma.

Ingredient 2

Lemma

There exists a k-edge cut S separating X from Y where X has precisely one high-degree vertex and the edges in S are vertex-disjoint.

Proof idea:

- there is a k-edge cut between any pair of high-degree vertices
- use submodularity of vertex degree
  - if $d(X_1) = k$ and $d(X_2) = k$, and $|X_1 \cup X_2| < n$, then $d(X_1 \cap X_2) = k$ by k-connectivity and submodularity
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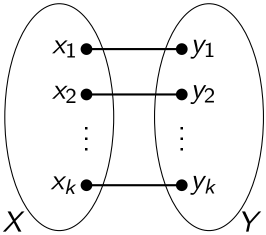
Ingredient 3

Lemma

G is not k-colourable if and only if for every k-colouring ϕ_X of $G[X]$ and every k-colouring ϕ_Y of $G[Y]$, the x_i's are all the same colour in ϕ_X and the y_i's are all different colours in ϕ_Y, or vice versa.



Proof:

- Construct an auxiliary graph from the colourings ϕ_X and ϕ_Y such that G is k-colourable iff the auxiliary graph is k-colourable
  - vertices are the colour classes of $\phi_X|\{x_1, \ldots, x_k\}$ and $\phi_Y|\{y_1, \ldots, y_k\}$
  - each set of colour classes is a clique
  - an edge between colour classes u and v if $\phi_X(x_i) = u$ and $\phi_Y(y_i) = v$ for some $i \in [k]$
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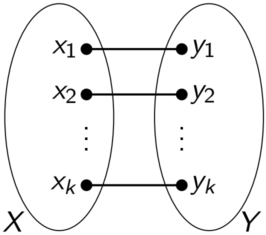

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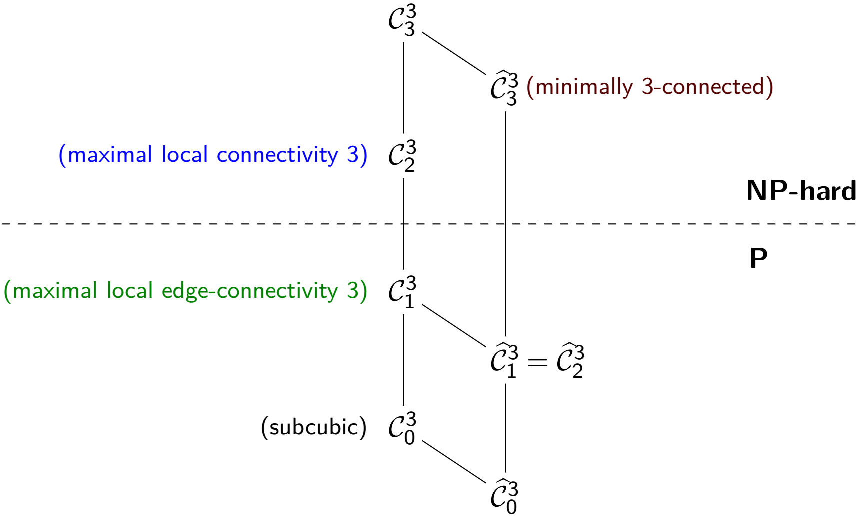
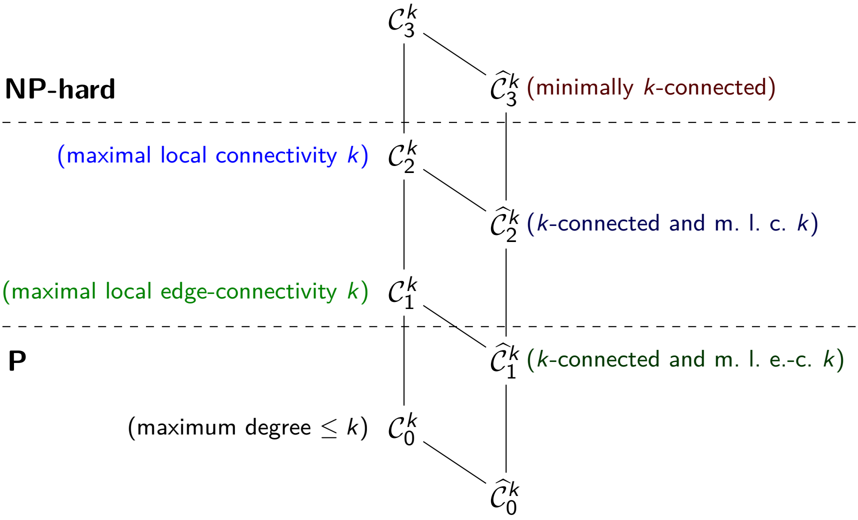
3-COLOURING complexity: results



k-COLOURING complexity, $k \geq 4$



Open questions

For fixed $k \geq 4$,

- is k-COLOURING, restricted to graphs with maximal local connectivity k, NP-hard?
- is k-COLOURING, restricted to k-connected graphs with maximal local connectivity k, in P?
- is k-COLOURING, restricted to graphs with maximal local edge-connectivity k, in P?

Thank you for your attention.