Parameterized Complexity Dichotomy for (r, ℓ)-Vertex Deletion

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Outline of the talk

1. (r, ℓ)-graphs
2. (r,ℓ)-Vertex Deletion
3. Independent (r,ℓ)-Vertex Deletion

Next section is...

1. (r, ℓ)-graphs
2. (r,ℓ)-Vertex Deletion
3. Independent (r,ℓ)-Vertex Deletion

Definition

An *(r,ℓ)-graph* is a graph whose vertex set can be partition into r independent sets and ℓ cliques.

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(1,0)-graphs Independent sets.

(0,1)-graphs Cliques.

(2,0)-graphs Bipartite graphs.

(1,1)-graphs Split graphs.

(r,0)-graphs r-colorable graphs.

Theorem

Let r and ℓ be two fixed integers. Let $G = (V, E)$ be a graph.

- *If $\max\{r, \ell\} < 3$ then we can check if G is an (r, ℓ)-graph and construct an (r, ℓ)-partition in polynomial time.*
- *Otherwise the recognition problem is NP-complete.*

[Brandstädt 96]

Next section is...

(r,ℓ)-VERTEX DELETION
Input: A graph $G = (V, E)$, an integer k.
Parameter: k.
Output: A set $S \subseteq V$ such that:

- $|S| \leq k$
- $G \backslash S$ is a (r, ℓ)-graph.

3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2				p-NP-c
1				p-NP-c
0				p-NP-c
ℓ \ r	0	1	2	3

3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2				p-NP-c
1				p-NP-c
0	P			p-NP-c
ℓ \ r	0	1	2	3

3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2				p-NP-c
1	$\overline{\text{VC}}$ 1.27^k			p-NP-c
0	P	VC 1.27^k		p-NP-c
ℓ \ r	0	1	2	3

[Chen, Kanj, Xia 10]

3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2	$\overline{\text{OCT}}$ 2.31^k			p-NP-c
1	$\overline{\text{VC}}$ 1.27^k			p-NP-c
0	P	VC 1.27^k	OCT 2.31^k	p-NP-c
ℓ \ r	0	1	2	3

[Reed, Smith, Vetta 04]

ℓ \ r	0	1	2	3
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[Foldes, Hammer 77]

3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2	$\overline{\text{OCT}}$ 2.31^k	NP-h	NP-h	p-NP-c
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3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
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[B., Faria, Klein, Sau on arXiv (abs/1504.05515) 21/04/2015]

[Kolay, Panolan on arXiv (abs/1504.08120) 30/04/2015]

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0	P	VC 1.27^k	OCT 2.31^k	p-NP-c
ℓ \ r	0	1	2	3

Theorem

There is no algorithm running in time $2^{o(k)} \cdot n^{O(1)}$ for solving (r,ℓ)-Vertex Deletion, for $r > 0$ or $\ell > 0$, unless the ETH fails.

$(2,1)$-Vertex Deletion
Input: A graph $G = (V, E)$, an integer k.
Parameter: k.
Output: A set $S \subseteq V$ such that:

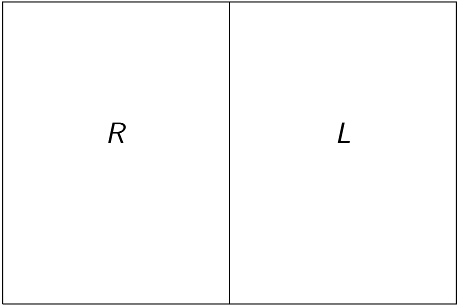
- $|S| \leq k$
- $G \backslash S$ is a $(2,1)$-graph.

Definition

Let $G = (V, E)$ be a graph. An (r, ℓ)-*partition* of G is a bipartition (R, L) of V such that R is a $(r, 0)$-graph and L is a $(0, \ell)$-graph.

Lemma

Let r and ℓ be two fixed integers, and let (R, L) and (R', L') be two (r, ℓ)-partitions of a graph G. Then we can find $L_{sel} \subseteq R$ and $R_{sel} \subseteq L$ both of size at most $r \cdot \ell$ such that $R' = (R \setminus L_{sel}) \cup R_{sel}$ and $L' = (L \setminus R_{sel}) \cup L_{sel}$.



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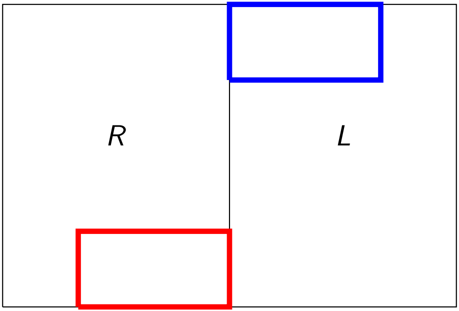

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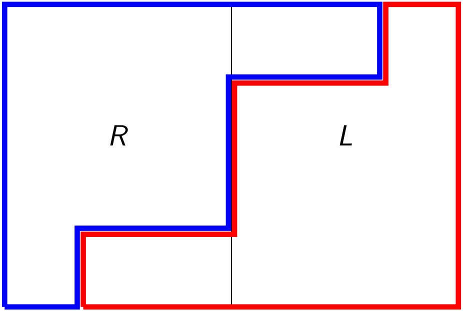

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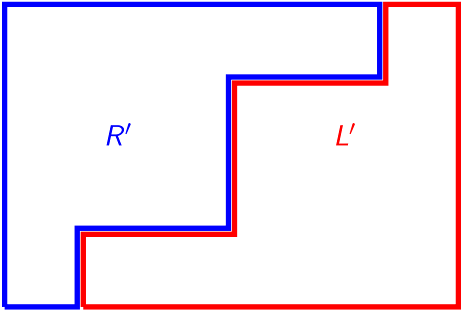

Lemma

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A similar lemma was proved by Feder, Hell, Klein, and Motwani in 2003.

DISJOINT $(2, 1)$-VERTEX DELETION
Input: A graph $G = (V, E)$, an integer k, and a set $S \subseteq V$ such that:

- $|S| \leq k + 1$
- $G \backslash S$ is a (2,1)-graph.

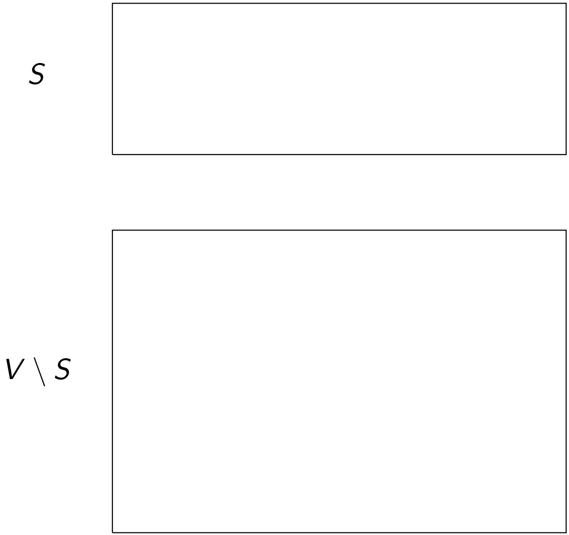
Parameter: k.
Output: A set $S' \subseteq V \setminus S$ such that:

- $|S'| \leq k$
- $G \backslash S'$ is a (2,1)-graph.

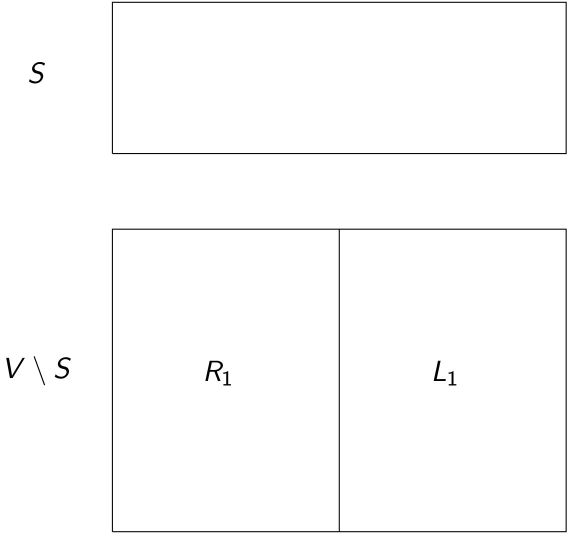
Lemma

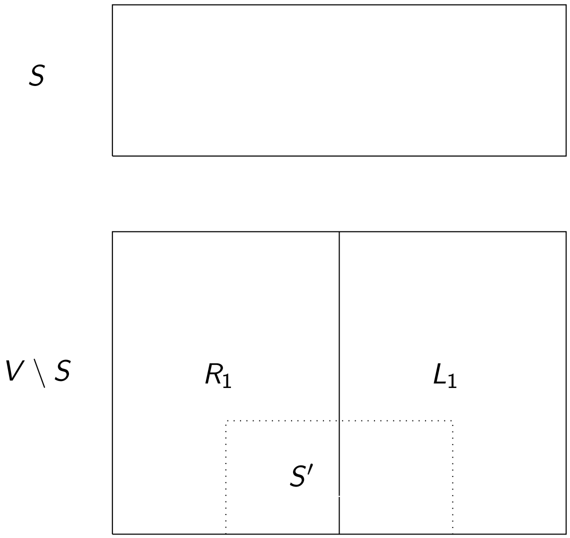
If Disjoint (2,1)-Vertex Deletion *can be solved in time* $c^k \cdot n^{\mathcal{O}(1)}$ *for some constant* c*, then* (2,1)-Vertex Deletion *can also be solved in time* $(c+1)^k \cdot n^{\mathcal{O}(1)}$.

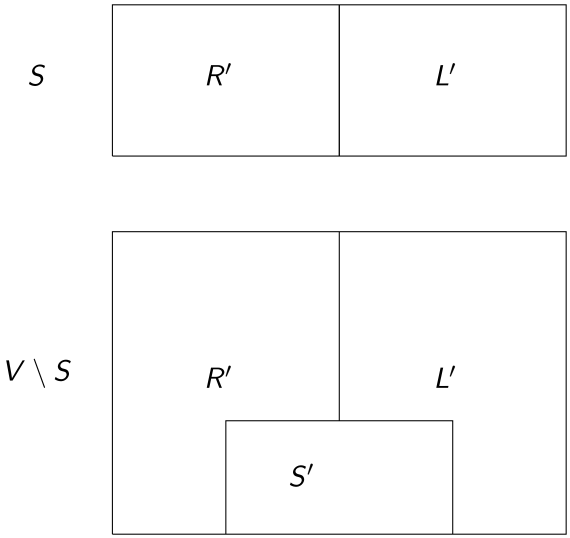
The iterative compression technique was introduced by Reed, Smith, and Vetta for the algorithm for Odd Cycle Transversal.

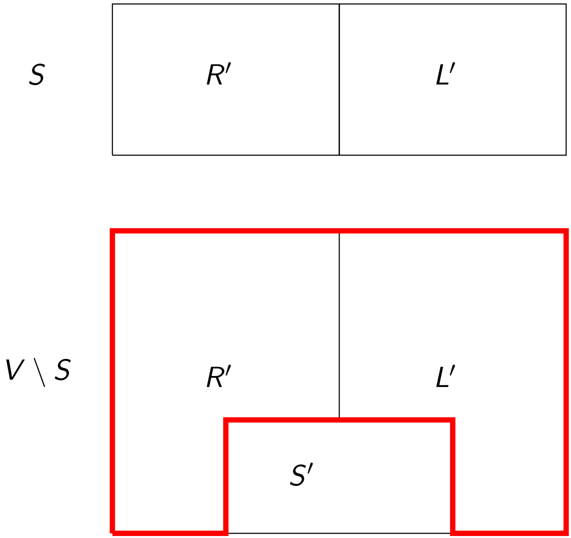

S
$V \setminus S$

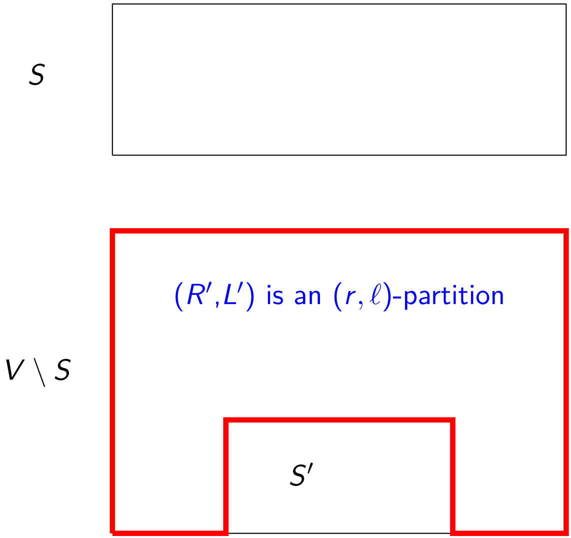

S
V \ S

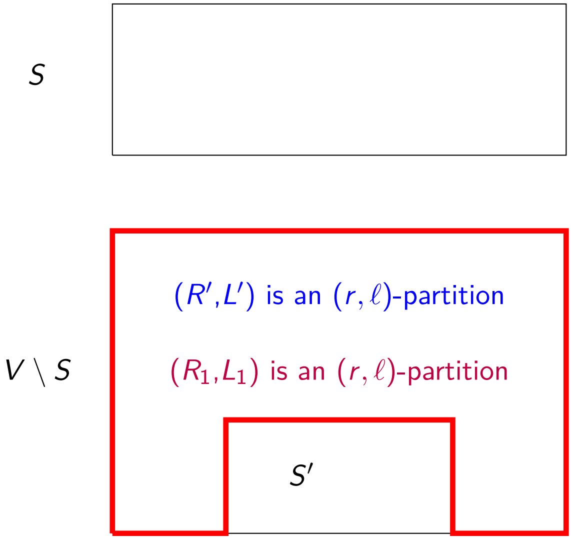

S
$V \setminus S$
R_1
L_1

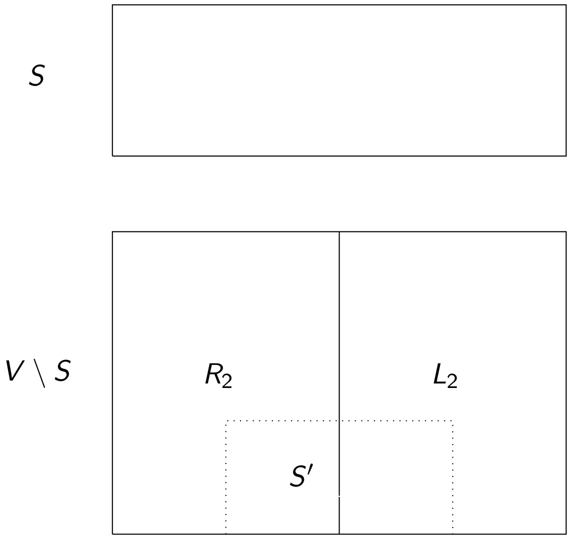

S
$V \setminus S$
R_1
L_1
S'


S
R'
L'
$V \setminus S$
R'
L'
S'

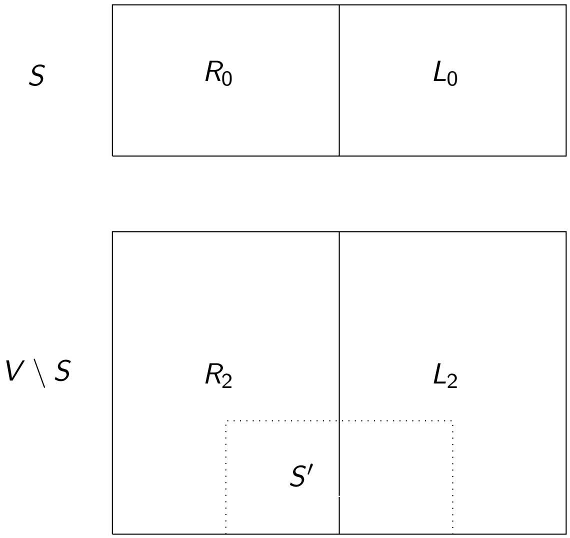

S
R'
L'
V \ S
R'
L'
S'


S
$V \setminus S$
(R', L') is an (r, ℓ)-partition
S'


S
$V \setminus S$
(R',L') is an (r,ℓ)-partition
(R_1,L_1) is an (r,ℓ)-partition
S'



Find (r, ℓ)-partitions
n^4



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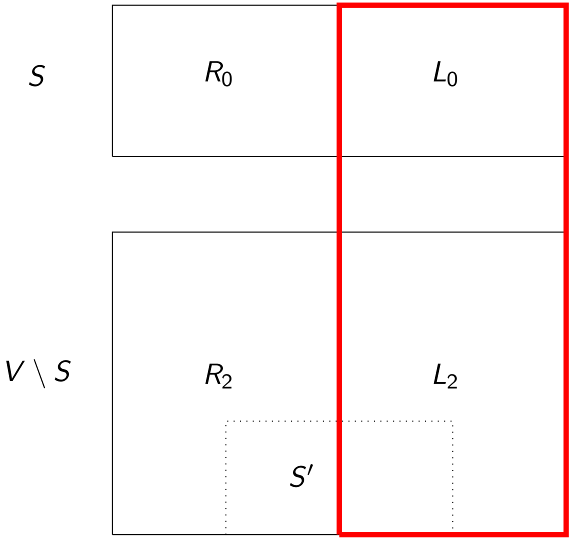

Find (r, ℓ)-partitions

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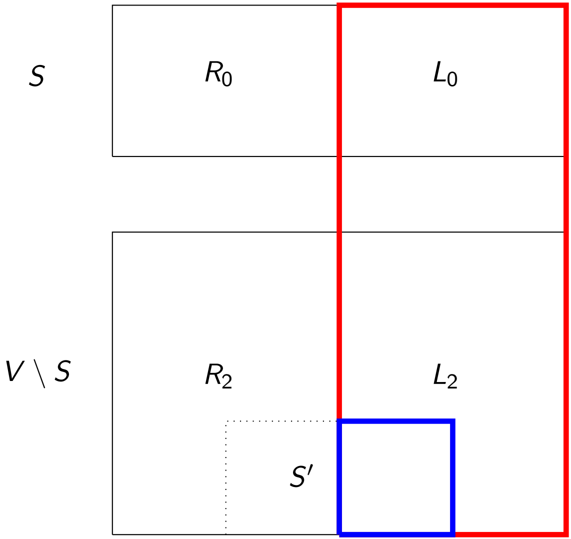
$\overline{VC}$

$1.27^k \cdot n^{\mathcal{O}(1)}$



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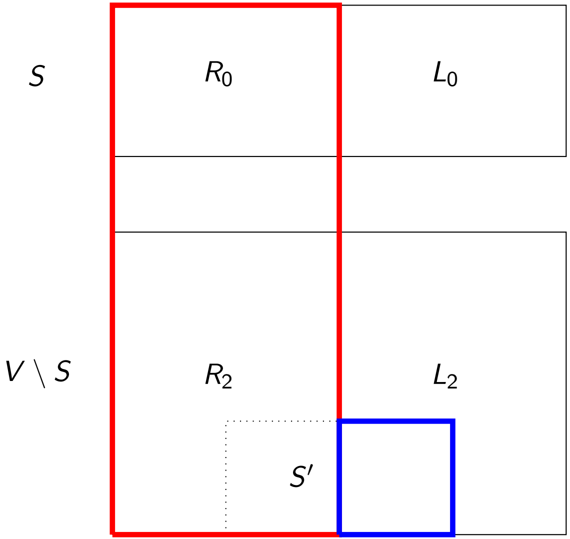
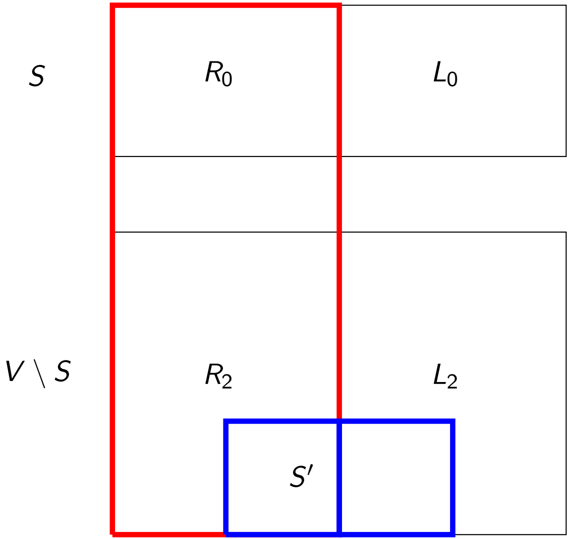


Find (r, ℓ)-partitions
n^4

$\overline{VC}$
$1.27^k \cdot n^{\mathcal{O}(1)}$

OCT
$2.31^k \cdot n^{\mathcal{O}(1)}$



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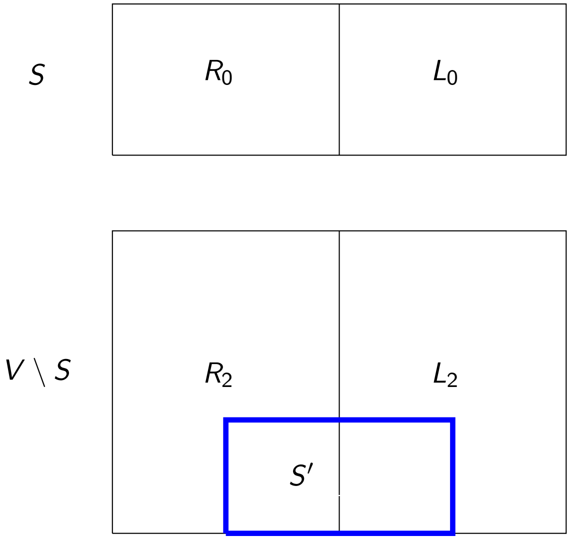

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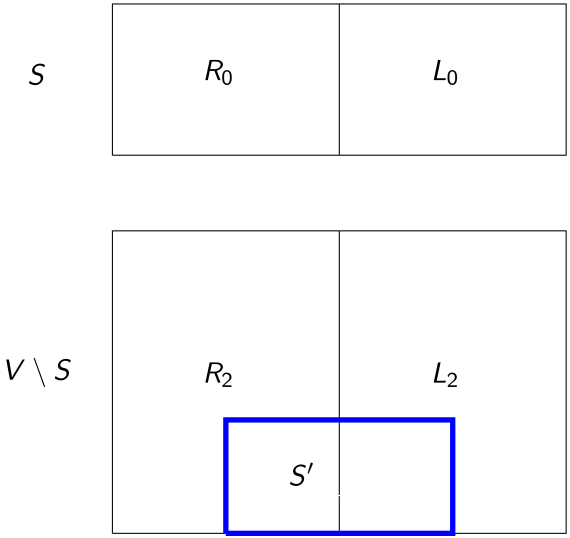

Find (r, ℓ)-partitions
n^4

$\overline{VC}$
$1.27^k \cdot n^{\mathcal{O}(1)}$

OCT
$2.31^k \cdot n^{\mathcal{O}(1)}$

$(2,1)$-VD
$3.31^k \cdot n^{\mathcal{O}(1)}$

Next section is...

1. (r, ℓ)-graphs
2. (r,ℓ)-Vertex Deletion
3. Independent (r,ℓ)-Vertex Deletion

Independent (r,ℓ)-Vertex Deletion
Input: A graph $G = (V, E)$, an integer k.
Parameter: k.
Output: An independent set $S \subseteq V$ of size at most k such that $G \backslash S$ is an (r,ℓ)-graph.

ℓ \ r	0	1	2	3
3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2				p-NP-c
1				p-NP-c
0				p-NP-c

3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2				p-NP-c
1				p-NP-c
0	P			p-NP-c
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3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2				p-NP-c
1				p-NP-c
0	P		IOCT $2^{2^{O(k^2)}}$	p-NP-c

[Marx, O'Sullivan, Razgon 13]

ℓ \ r	0	1	2	3
3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2	P	P	NP-h	p-NP-c
1	P	P	NP-h	p-NP-c
0	P	IVC P	IOCT $2^{2^{O(k^2)}}$	p-NP-c

ℓ \ r	0	1	2	3
3	p-NP-c	p-NP-c	p-NP-c	p-NP-c
2	P	P	$2^{2^{O(k^2)}}$	p-NP-c
1	P	P	$2^{2^{O(k^2)}}$	p-NP-c
0	P	IVC P	IOCT $2^{2^{O(k^2)}}$	p-NP-c

Theorem

Independent Odd Cycle Transversal *is FPT when parameterized by the size of the solution.*

[Marx, O'sullivan, Razgon 15]

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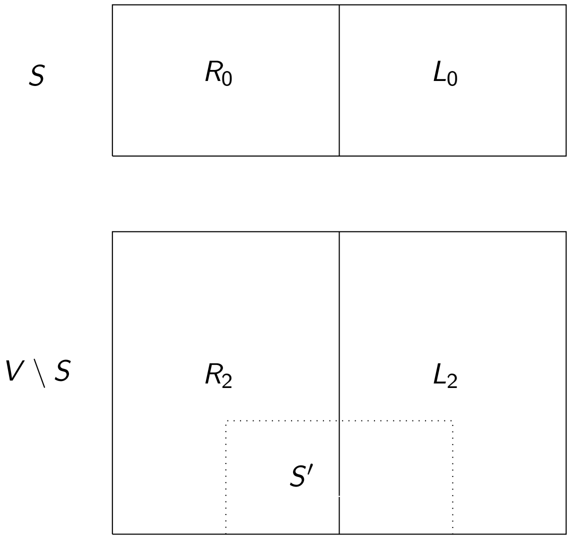
[Marx, O'sullivan, Razgon 15]

Theorem

Independent Odd Cycle Transversal *can be solved in* $2^{2^{\mathcal{O}(k^2)}} \cdot n^{\mathcal{O}(1)}$ *where k is the size of the solution.*

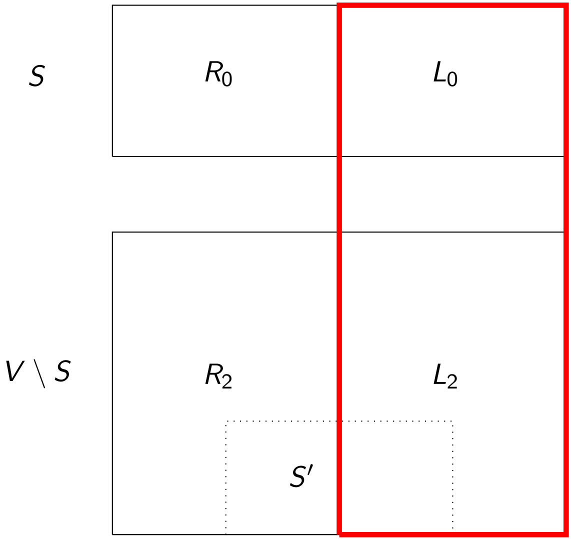
Theorem

Independent $(2, 1)$-vertex Deletion *and* Independent $(2, 2)$-Vertex Deletion *are FPT.*



Find (r, ℓ)-partitions

n^4

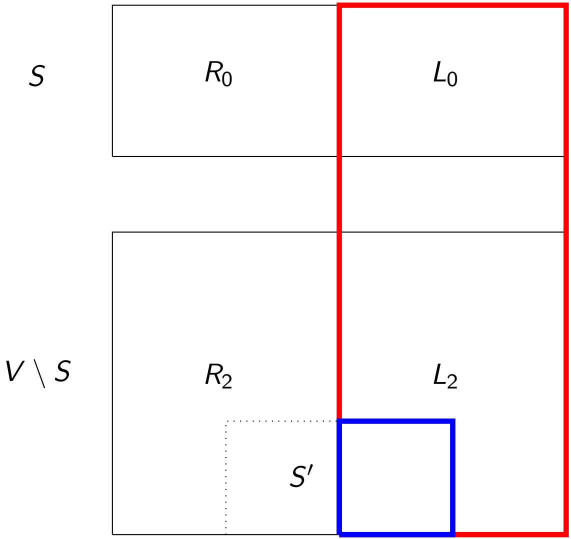


Find (r, ℓ)-partitions

n^4

$L_0 \cup L_2$ be a clique

n



Find (r, ℓ)-partitions

n^4

$L_0 \cup L_2$ be a clique

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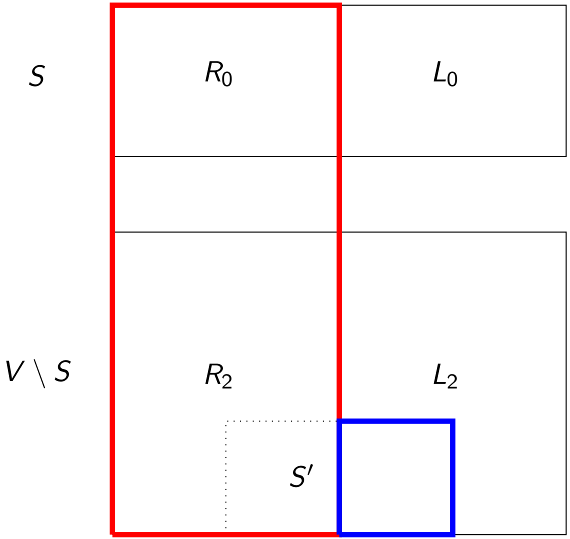


Find (r, ℓ)-partitions

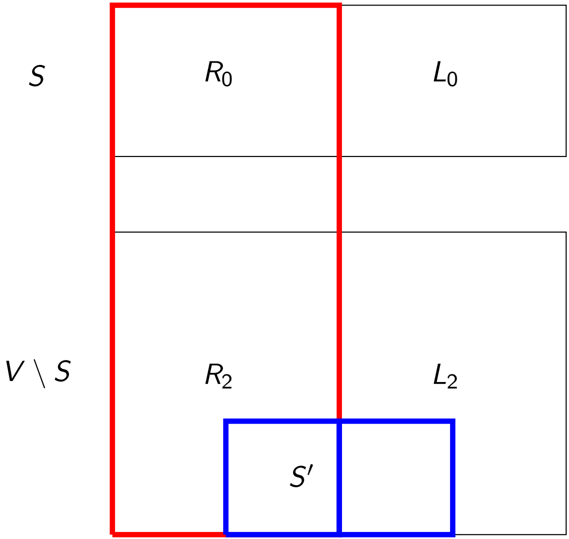
n^4

$L_0 \cup L_2$ be a clique

n

IOCT

$2^{2^{\mathcal{O}(k^2)}} \cdot n^{\mathcal{O}(1)}$



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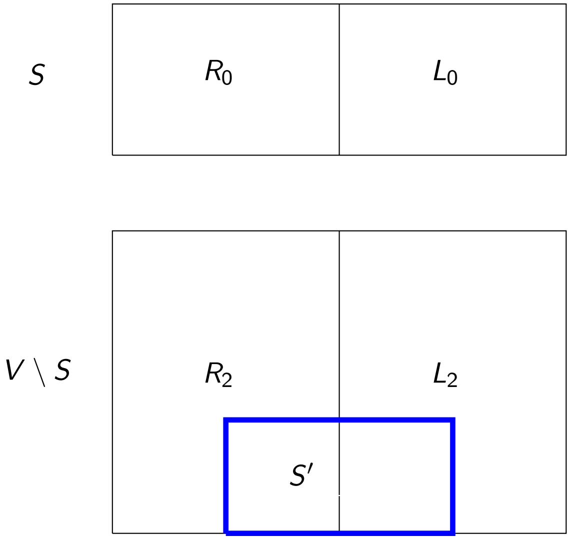

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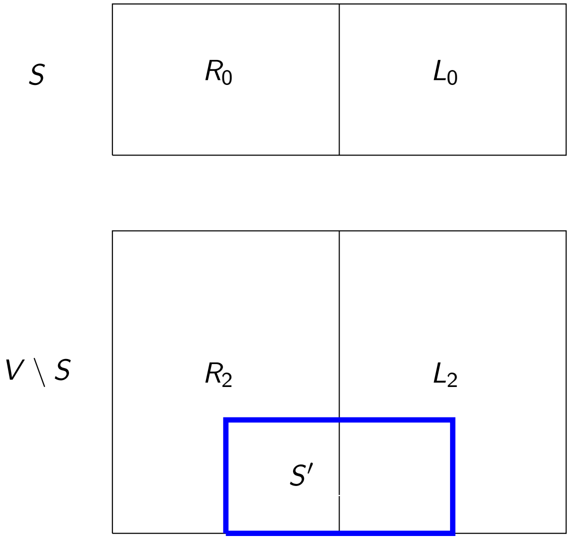

Find (r, ℓ)-partitions

n^4

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n

IOCT

$2^{2^{\mathcal{O}(k^2)}} \cdot n^{\mathcal{O}(1)}$

I(2, 1)-VD

$2^{2^{\mathcal{O}(k^2)}} \cdot n^{\mathcal{O}(1)}$

Further research

- Can we improve the running time for Independent (r, ℓ)-Vertex Deletion?

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- Does a polynomial kernel for (r, ℓ)-Vertex Deletion exist?
  - There is a randomized polynomial kernel for Odd Cycle Transversal using matroids. [Kratsch, Wahlström 14]

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- Can we improve the running time for Independent (r, ℓ)-Vertex Deletion?
- Does a polynomial kernel for (r, ℓ)-Vertex Deletion exist?
  - There is a randomized polynomial kernel for Odd Cycle Transversal using matroids. [Kratsch, Wahlström 14]
- Is $(2, 2)$-Edge Deletion FPT?
  - $(2, 1)$-Edge Deletion and $(1, 2)$-Edge Deletion are FPT. [Kolay, Panolan 15]

Thanks