Cutwidth in Tournament and Applications

GROW

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Thursday 15^{th} October 2015

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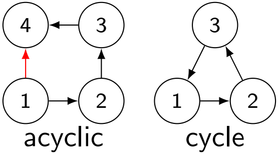

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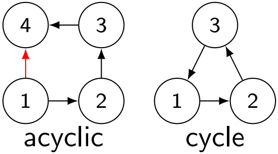

Does there exist a better WQO relation for digraphs?

	Undirected Graph	Semi-Complete Digraph	
WQO Relation	Graph Minor	Digraph Minor	Immersion
Width Measure	Treewidth	Pathwidth	Cutwidth

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During this talk, we focus on *Immersion* and *Cutwidth*.

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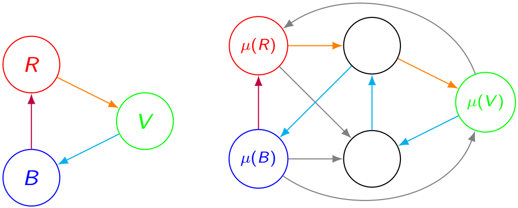
An **immersion** μ of H in D satisfies

- $\forall u \in V_H$, $\mu(u)$ is a distinct vertex of V_D *Vertex injection*



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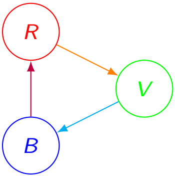

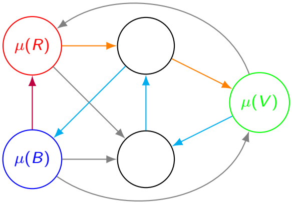

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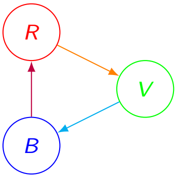
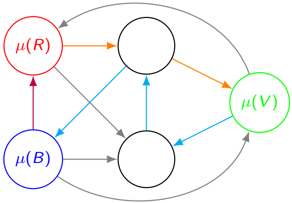
- $\forall uv, u'v' \in E_H, \mu(uv) \cap \mu(u'v') = \emptyset$ — *Disjoint paths*



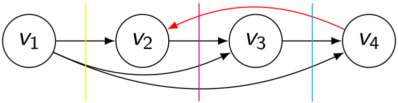


Immersion

Vertex ordering and cuts

$\pi = (v_{\pi(1)}, \dots, v_{\pi(n)})$ is an ordering of V.
$\pi[i]$ contains the first i vertices of π

- The i^{th} **cut of** π is: $E_\pi[i] = \{uv \in E \mid u \in V \setminus \pi[i] \text{ and } v \in \pi[i]\}$



$E_{id}[2] = E_{id}[3] = \{v_4 v_2\}$

	v_1	v_2	v_3	v_4
v_1	-	0	0	0
v_2	1	-	0	1
v_3	1	1	-	0
v_4	1	0	1	-

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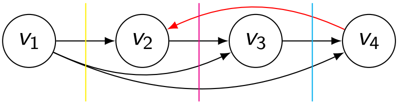
- $uv \in E_\pi[i]$ is a **backward arc** for π since $v <_\pi u$.



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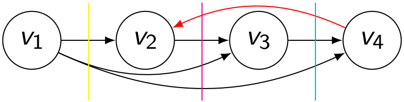
Cutwidth

Cutwidth of π: $cw(D,\pi) = \max_{0 \leq i \leq n} |E_\pi[i]|$

Cutwidth of D: $cw(D) = \min_{\pi} cw(D,\pi)$



$cw(D, id) = 1$

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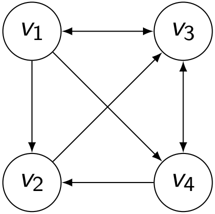
Some Properties

- Cutwidth-0 = Directed Acyclic Graph
- Cutwidth $\in$ NP-complete
- Cutwidth is closed under Immersion

Semi-complete digraphs and tournaments

D is **semi-complete** if:

- it is simple (no self-loop, no multiple arc)
- for every distinct $u, v \in V$, uv or vu is in E.



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Semi-complete digraphs and tournaments

D is a **tournament** if:

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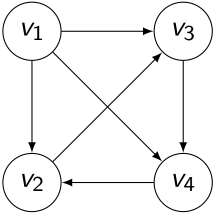

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Strong known results

Chudnovsky and Seymour 2011

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Chudnovsky, Fradkin and Seymour 2012

Given a fixed digraph H and given a semi-complete D as input, testing whether there exists an immersion of H in D is FPT with parameter $|V(H)|$.

Cutwidth in Semi-Complete

$$\pi \in V^- \text{ if } \forall u, v \in V, d^-(u) < d^-(v) \Rightarrow u <_\pi v.$$

Alon, Lokshtanov and Saurabh 2009; Pilipczuk 2013

- $\pi \in V^- \Rightarrow cw(D, \pi) \leq 16cw(D)^2 + 10cw(D) + 1$
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Cutwidth in Semi-Complete $\in$ NP-Hard?
And-Composition + NP-Hard $\Rightarrow$ no polynomial kernel

Linear approximation?

Cutwidth in Tournament

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My Observation

- $\pi \in V^- \Rightarrow cw(T, \pi) = cw(T)$
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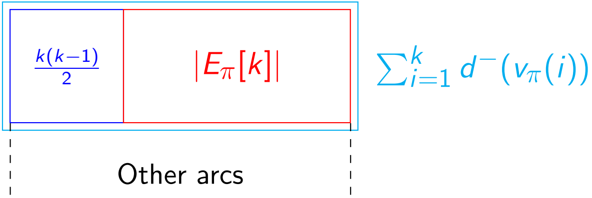
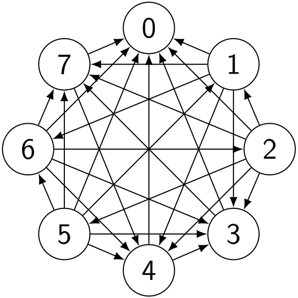


Example



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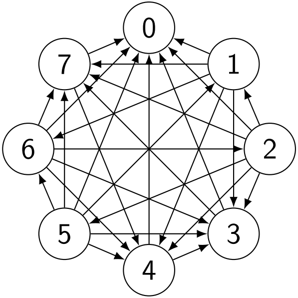

i	0	1	2	3	4	5	6	7
$\pi(i)$	2	5	1	6	4	3	7	0
$d^-(v_{\pi(i)})$	1	1	2	2	5	5	5	7
$d^-(v_{\pi(i)}) - i$	1	0	0	-1	1	0	-1	0
$\lvert E_\pi(i)\rvert$	1	1	1	0	1	1	0	0

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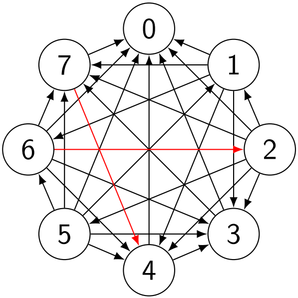

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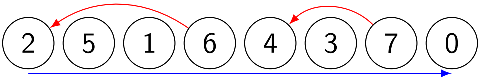

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c-Certificate

A *c-certificate* is a couple (L, R) of disjoint vertex sets s.t.:

- $\max\{|L|, |R|\} \leq c \leq E(R, L)$
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$cw(T) \geq c \Leftrightarrow T$ contains a c-certificate

c-Obstacles

Lemma : existence of small obtacles

If $cw(T) \geq c$, T contains an induced subtournament T' s.t.:

- $cw(T') \geq c$; same c-certificate
- $|V_{T'}| = O(c^2)$; all constants are small

Moreover, one can find a c-obstacle $T' \prec_s T$ in polynomial time.

1-obstacle = triangle; all 2-obstacles have 5 vertices.

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Conjecture : $|V_{T'}| = O(c)$

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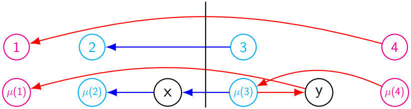
Note:

- 0-CVD = Feedback Vertex Deletion
- 0-CAR = Feedback Arc Set
- CAR $\Leftrightarrow$ Cutwidth Arc Deletion + keep semi-complete structure.

FPT algorithm for Semi-Complete

c-CAR is closed under immersion



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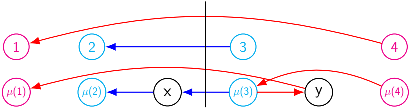

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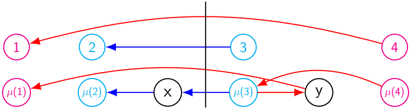

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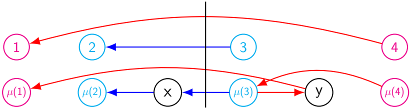

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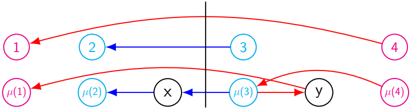

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  $O^*((c+k)^{O(c+k)})$-time dynamic programming for c-CARS.

Branching Rules for c-CVD in Tournament

If T contains a c-obstacle T', $\forall u \in V_{T'}$ branch on $(T/u, k-1)$.

c-Cutwidth Vertex Deletion in Tournament admits an $O^*(c^{2k})$-time branching algorithm.

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Branching Rules for c-CAR in Tournament

If T contains a c-obstacle T' with c-certificate (L, R), $\forall uv \in E_{T'}$ s.t. $\{u, v\} \cap (L \cup R) \neq \emptyset$, branch on $(T[\overrightarrow{\{uv\}}], k-1)$.

c-Cutwidth Arc Reversal in Tournament admits an $O^*(c^{3k})$-time branching algorithm.

Further Works

- Cutwidth Complexity:
  - NP-Hardness for semi-complete digraphs?
  - Linear Approximation?
- Improving what we have already done:
  - $O(c)$-sized c-obstacles? c-obstacles for semi-complete?
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  - Similar results for Pathwidth and Digraph Minor?

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Thanks for your attention.