Mixed structures in digraphs and completing orientations of partially oriented graphs

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GROW 2015, Aussois, France

[1] Based on joint works with Carl Johan Casselgren, Jing Huang, Matthias Kriesell, Alessandro Maddaloni, Saket Saurabh, Sven Simonsen, Anders Yeo, and Xuding Zhu

Edge disjoint spanning trees in graphs

Theorem (Tutte 1961, Nash-Williams 1961)

A graph G contains k edge-disjoint spanning trees if and only if

$$|E_G(\mathcal{P})| \geq k \cdot (|\mathcal{P}| - 1)$$

holds for all partitions $\mathcal{P}$ of $V(G)$.

$E_G(\mathcal{P})$: set of edges in G between distinct parts of $\mathcal{P}$.

Arc disjoint branchings in digraphs

Theorem (Edmonds 1973)

For a vertex r of a digraph D there exists k arc-disjoint branchings with root r if and only if

$$d^+(X) \geq k$$

for every proper subset X of V(D) containing r.

$d^+(X)$: number of arcs in D from some $x \in X$ to some $y \in V(D) - X$.

An intermediate problem

Problem (Thomassé, Egres Open Problems List 2008)

Find a good characterization of the digraphs D such that there exist edge-disjoint S, T, where S is a spanning tree of $UG(D)$ and T is an out-branching of D.

$UG(D)$: underlying undirected graph; technically:
same vertices and edges, different incidence relation

Obv. necessary: two edge-disjoint spanning trees in $UG(D)$.
Obv. sufficient: two arc-disjoint out-branchings in D.

A problem on mixed paths

Let D be a digraph and $r \in V(D)$.

If there are edge-disjoint S, T, where
S is a spanning tree of $UG(D)$ and
T is an out-branching of D rooted at r
then for each $s \in V(D)$ there exist edge-disjoint P, Q where
P is an (r, s)-path in $UG(D)$ and Q is an (r, s)-path in D.

Problem (Mixed-edge-disjoint-paths)

*Given a digraph D and $r, s \in V(D)$,
decide if there exist edge-disjoint P, Q, where
P is an (r, s)-path in $UG(D)$ and Q is an (r, s)-path in D.*

Obv. necessary: two edge-disjoint (r, s)-paths in $UG(D)$.
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Obv. sufficient: two arc-disjoint (r,s)-paths in D.

The unmixed versions

Theorem (Menger 1927)

Given two vertices $r \neq s$ of a graph or digraph D, there exist k edge-disjoint (r,s)-paths if and only if there no (r,s)-cut X with $|X| < k$ in D.

X an (r,s)-cut: every (r,s)-path in D contains an arc from X.

The intermediate version is difficult

Theorem (Bang-Jensen & Kriesell 2009)

MIXED-EDGE-DISJOINT-PATHS *is* $\mathcal{NP}$*-complete.*

Mixed homeomorphisms

Let H be a fixed mixed graph and D be any digraph.

A *mixed homeomorphism f from H into D* maps

- each vertex of H to a vertex of D,
- each directed edge xy to a nontrivial $(f(x), f(y))$-path in D, and
- each undirected edge xy to a nontrivial $(f(x), f(y))$-path in $UG(D)$

such that

- $f(x) \neq f(x')$ for $x \neq x'$ in $V(H)$ and
- $Int(f(e)) \cap f(e') = \emptyset$ for $e \neq e'$ in $E(H)$.

In this definition, a cycle through $f(x)$ is considered as a nontrivial $(f(x), f(x))$-path with end vertex $f(x)$ in D or $UG(D)$. $Int(f(e))$ is the set of all vertices of $f(e)$ except its end(s). Homeomorphisms from *graphs* into *graphs* are defined accordingly.

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A class of homeomorphism extension problems.

Fix a mixed graph H.

Problem (MIXED-HOMEOMORPHISM-EXTENSION)

Given a digraph D and an injection $f : V(H) \to V(D)$, decide if f extends to a mixed homeomorphism from H into D.

Roughly, we look for a subdivision of H in D, where we fix the principal vertices and do not care about the direction of the edges of the subdivision paths or cycles corresponding to undirected edges of H.

The undirected case

Fix a graph H.

Problem (HOMEOMORPHISM-EXTENSION)

Given a graph G and an injection $f : V(H) \to V(G)$, decide if f extends to a homeomorphism from H into G.

To solve this, it is sufficient to solve polynomially many linkage problems. Each of these takes polynomial time by Graph Minors XIII (Robertson & Seymour 1995).

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The directed case

Fix a digraph H.

Problem (DIGRAPH-HOMEOMORPHISM-EXTENSION)

Given a digraph D and an injection $f : V(H) \to V(D)$, decide if f extends to a [mixed] homeomorphism from H to D.

There is a classic dichotomy stating that DIGRAPH-HOMEOMORPHISM-EXTENSION is solvable in polynomial time if all edges of H have the same initial vertex or they all have the same terminal vertex, whereas, in all other cases, it is $\mathcal{NP}$-complete (Fortune & Hopcroft & Wyllie 1980).

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The general dichotomy

Theorem (Bang-Jensen & Kriesell 2009)

MIXED-HOMEOMORPHISM-EXTENSION *for H is in $\mathcal{P}$ if*

- *all edges of H are undirected, or*
- *all edges of H are directed and all have the same initial vertex or all have the same terminal vertex,*

and it is $\mathcal{NP}$-complete in all other cases.

The case that H consists of a directed and an undirected loop at distinct vertices can be rephrased:

Problem

Given a digraph D and vertices $x \neq y$, decide if there is a cycle B in D and a cycle C in $UG(D)$ such that $x \in V(B)$, $y \in V(C)$, and $V(B) \cap V(C) = \emptyset$.

The problem is $\mathcal{NP}$-complete — even if we do not prescribe y.

It is likely that this changes if we do neither prescribe y nor x.

The special case of two cycles

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Disjoint cycles in graphs and digraphs

Problem (DISJOINT-CYCLES)

Decide if a given (di)graph G has two disjoint cycles.

In $\mathcal{P}$ for graphs by classic results (Lovász 1965, Dirac 1963).

In $\mathcal{P}$ for directed graphs (McCuaig 1993); difficult.

The mixed disjoint cycles problem

Problem (Mixed-disjoint-cycles)

Decide if, for a given digraph, there exists cycles B in D and C in $UG(D)$ such that $V(B) \cap V(C) = \emptyset$.

Theorem (Bang-Jensen & Kriesell 2009)

Mixed-disjoint-cycles *restricted to strongly connected digraphs D is in $\mathcal{P}$, and B, C as there can be found in polynomial time if they exist.*

A vertex v of a non acyclic digraph D is a **transversal vertex** if $D - v$ is acyclic.

If $v_1, v_2, \ldots, v_k$ are transversal vertices of D, then they occur in the same order on all dicycles.

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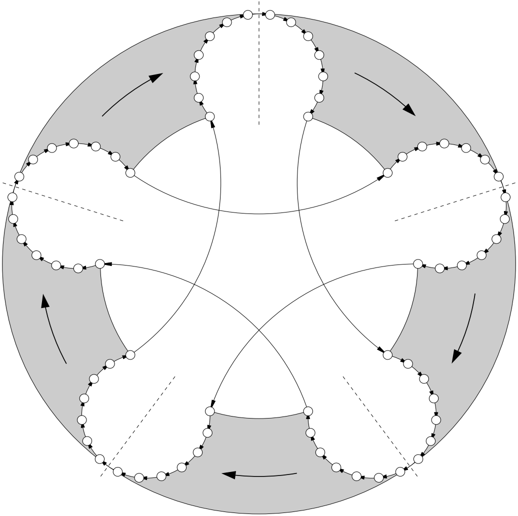
Theorem (Bang-Jensen, Kriesell, Maddaloni & Simonsen, 2014)

For non-strong digraphs with a bounded number of transversal vertices MIXED-DISJOINT-CYCLES *is in $\mathcal{P}$. Without this restriction the problem is $\mathcal{NPC}$.*

A vault



A strong digraph D with $\tau(D) = 2$ and no pair of disjoint cycle, dicycle.

Arc-disjoint cycle and dicycle problem

Problem (MIXED-ARC-DISJOINT-CYCLES)

Decide if, for a given digraph, there exists cycles B in D and C in $UG(D)$ such that $A(B) \cap A(C) = \emptyset$.

Observation: If a digraph D does not contain a dicycle B and a cycle C in $UG(D)$ which are arc-disjoint then for every dicycle B, $D - A(B)$ is an oriented forrest.

Theorem (Bang-Jensen, Kriesell, Maddaloni & Simonsen, 2015)

For strong digraphs and non-strong digraphs with a bounded number of transversal arcs MIXED-ARC-DISJOINT-CYCLES *is in $\mathcal{P}$. Without this restriction the problem is $\mathcal{NPC}$.*

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Mixed cycle-factors in digraphs

Problem (MIXED CYCLE-FACTOR)

Given a digraph D; does $UG(D)$ contain a 2-factor $C_1, C_2, \ldots, C_k$ so that C_1 is a directed cycle in D?

The non-mixed versions 2-factor in graphs and cycle-factor in digraphs are well-known polynomial problems.

Theorem (Bang-Jensen & Casselgren, 2015)

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Solution to Thomassé's problem

Theorem (Bang-Jensen and Yeo 2010)

The following problem is NP-complete: Given a directed graph $D = (V, A)$ and a vertex $s \in V$; does D have an out-branching B_s^+ such that $UG(D - A(B_s^+))$ is connected?

Sketch of proof:

First step: reduce 3-SAT to (s, t)-path in a digraph which avoids certain vertices.

Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. The ordering of the clauses $C_1, C_2, \ldots, C_m$ induces an ordering of the occurrences of a variable x and its negation $\bar{x}$ in these.

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Let $W[u, v, p, q]$ be the digraph (the variable gadget) with vertices $\{u, v, y_1, y_2, \dots y_p, z_1, z_2, \dots z_q\}$ and the arcs of the two (u, v)-paths $uy_1y_2 \dots y_pv, uz_1z_2 \dots z_qv$.

With each variable x_i we associate a copy of $W[u_i, v_i, p_i, q_i]$ where x_i occurs p_i times and $\bar{x}_i$ occurs q_i times in the clauses of $\mathcal{F}$. Identify end vertices of these digraphs by setting $v_i = u_{i+1}$ for $i = 1, 2, \dots, n-1$. Let $s = u_1$ and $t = v_n$. Denote the resulting digraph by D'.

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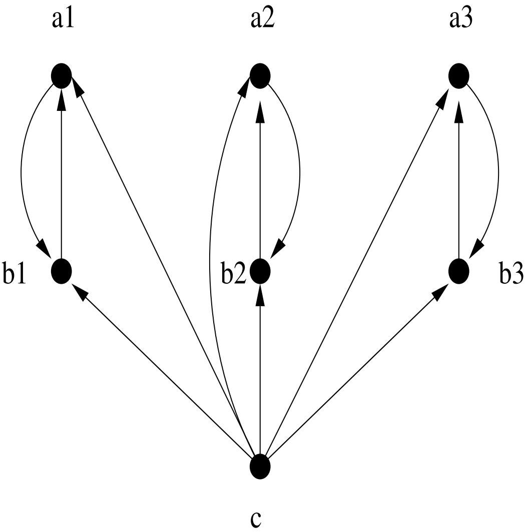
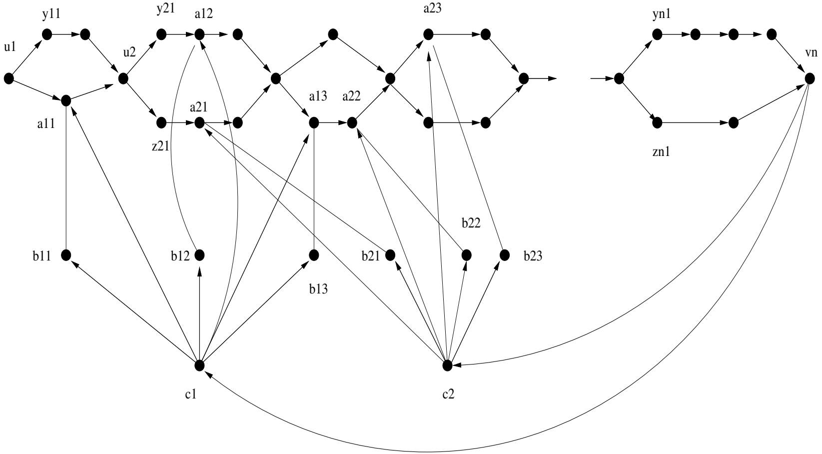
For each clause $C_j = \{a_{j,1} \vee a_{j,2} \vee a_{j,3}\}$ we identify $a_{j,i}$, $i = 1, 2, 3$ with the vertex corresponding to that litteral in D'.

Easy observation: D' contains an (s, t)-path P which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$ if and only if $\mathcal{F}$ is satisfiable.



The clause gadget.



A schematic picture of $D_{\mathcal{F}}$. Only the chain of variable gadgets and the clause gadgets corresponding to $C_1 = (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$ and $C_2 = (\bar{x}_2 \vee \bar{x}_3 \vee x_4)$ are shown

Claim: $D_{\mathcal{F}}$ has an out-branching B_s^+ such that $D_{\mathcal{F}} - A(B_s^+)$ is connected if and only if D' contains an (s,t)-path P which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$.

Suppose first that there exists B_s^+ such that $D - A(B_s^+)$ is connected. It follows from the structure of $D_{\mathcal{F}}$ that the (s,t)-path P in B_s^+ lies entirely inside D' and since tc_i is the only arc entering c_i, all arcs of the form tc_i, $i \in [m]$ are in B_s^+.

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P cannot contain all of $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for some clause C_j because that would disconnect the vertices of $H_j - \{a_{j,1}, a_{j,2}, a_{j,3}\}$ from the remaining vertices.

Conversely, suppose that D' contains an (s, t)-path P which avoids at least one vertex from $\{a_{j,1}, a_{j,2}, a_{j,3}\}$ for each $j \in [m]$. Then we form an out-branching B_s^+ by adding the following arcs:

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It remains to show that $D^* = D_{\mathcal{F}} - A(B_s^+)$ is connected. First observe that $D^*\langle V(D')\rangle$ contains either all arcs of the subpath $u_i y_{i,1} y_{i,2} \ldots y_{i,p_i} v_i$ or all arcs of the subpath $u_i z_{i,1} z_{i,2} \ldots z_{i,q_i} v_i$ for each $i \in [n]$ and hence it contains an (s,t)-path which passes through all the vertices $u_1, u_2, \ldots, u_n, t$.

By the description of P above, for each clause C_j, $j \in [m]$ and $i \in [3]$, if P contains the vertex $a_{j,i}$ then D^* contains the arcs $c_j b_{j,i}, c_j a_{j,i}$ and if P does not contain the vertex $a_{j,i}$ then D^* contains the arcs $c_j a_{j,i}, a_{j,i} b_{j,i}$. Now it is easy to see that D^* is connected and spanning.

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Further hardness results

Theorem (Bang-Jensen & Yeo, 2010)

It is $\mathcal{NP}$-complete to decide whether a given digraph has an (s,t)-path P such that $D - A(P)$ is connected for specified vertices s,t. □

Theorem (Bang-Jensen & Yeo, 2010)

It is $\mathcal{NP}$-complete to decide for a given digraph D and distinct vertices vertex $s,t \in V(D)$ whether the underlying graph of D has an (s,t)-path Q such that $D - A(Q)$ has an out-branching B_s^+ rooted at s.

Theorem (Bang-Jensen & Yeo, 2010)

It is $\mathcal{NP}$-complete to decide for a given strongly connected digraph D whether D contains a directed cycle C such that $UG(D - A(C))$ is connected.

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It is $\mathcal{NP}$-complete to decide for a given strongly connected digraph D whether $UG(D)$ contains a cycle C such that $D - A(C)$ is strongly connected.

Theorem (Bang-Jensen & Simonsen, 2013)

It is $\mathcal{NP}$-complete to decide whether a 2-regular digraph D contains a spanning strong subdigraph D' such that $UG(D - A(D''))$ is connected.

FPT algorithm for non disconnecting out-branchings

Problem (NON-DISCONNECTING OUT-BRANCHING)

Given a digraph D and a natural number k; does D have an out-branching B_s^+ and a spanning tree T such that $|A(B_s^+) - A(T)| \geq k$?

Theorem (Bang-Jensen, Saurabh & Simonsen, 2015)

The problem NON-DISCONNECTING OUT-BRANCHING *is fixed parameter tractable and admits a linear vertex kernel.*

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The orientation completion problem

A **partially oriented graph (a pog)** $P = (V, E \cup A)$ is a mixed graph consisting of both edges and arcs (possibly $E = \emptyset$ or $A = \emptyset$).

By **completing** the orientation of P we mean assigning an orientation to each edge $e \in E$.

Let $\mathcal{C}$ be a given class of digraphs (e.g. tournament, acyclic, strong, ...).

Problem ($\mathcal{C}$-ORIENTATION-COMPLETION PROBLEM)

Given a pog $P = (V, E \cup A)$; can we complete the orientation so that the resulting oriented graph D belongs to $\mathcal{C}$?

Common generalization of the recognition problem for $\mathcal{C}$ and the problem of deciding whether a graph is the underlying graph of some digraph from $\mathcal{C}$.

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Some known results

For a given pog $P = (V, E \cup A)$ we denote by $\overleftrightarrow{P}$ the digraph that we obtain by replacing each edge $uv \in E$ by a directed 2-cycle.

Theorem (Boesch & Tindell, 1980)

A partially oriented graph $P = (V, E \cup A)$ can be completed into a strongly connected oriented graph D if and only if $\overleftrightarrow{P}$ is strongly connected and $UG(P)$ has no bridge. This can be decided, and a strong orientation found when one exists, in polynomial time.

Theorem (Fekete, Köhler & Teich, 2000)

The $\mathcal{C}$-ORIENTATION-COMPLETION PROBLEM is polynomially solvable when $\mathcal{C}$ is the class of transitive oriented graphs.

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Completing to get an acyclic digraph with an (s, t)-path

Problem ((s,t)-PATH COMPLETION IN ACYCLIC DIGRAPH)

Input: a partially oriented graph $P = (V, E \cup A)$ *with two prescibed vertices* s, t. *Question: Does there exist a completion of* P *to an acyclic digraph with an* (s, t)*-path?*

Theorem (Bang-Jensen and Kriesell, 2015)

Problem (s,t)-PATH COMPLETION IN ACYCLIC DIGRAPH *is NP-complete.*

A **tournament** is an orientation of a complete graph. A digraph is **semicomplete** if every pair of distinct vertices is joined by an arc or by two arcs which form a directed 2-cycle.

A digraph D is **locally semicomplete** if $D[N^+(v)], D[N^-(v)]$ are semicomplete for every vertex v. D is a **local tournament** if $D[N^+(v)], D[N^-(v)]$ are tournaments for all vertices v.

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A **proper circular arc graph** is a graph which is the intersection graph of a family of circular arcs on a circle so that no such interval is properly contained in any other.

Theorem (Skrien, 1982)

Let G be a connected graph. The following statements are equivalent:

(a) *G can be completed to a local tournament.*
(b) *G can be completed to a locally transitive local tournament.*
(c) *G is a proper circular arc graph.*

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A **proper interval graph** is a graph which is the intersection graph of a family of intervals on the real line so that no such interval is properly contained in any other.

Theorem (Hell & Huang, 1995)

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Orienting a graph as a local tournament

Let $G = (V, E)$ be a graph. The **auxiliary graph** G^+ of G is defined as follows. The vertex set of G^+ consists of all ordered pairs (u, v) for all $uv \in E$ (note that every edge of G gives rise to two vertices of G^+). Two vertices (u, v) and (u', v') of G^+ are adjacent if and only if one of the following conditions holds:

- $u = u'$ and $vv' \notin E$;
- $uu' \notin E$ and $v = v'$;
- $u = v'$ and $v = u'$.

Lemma (Huang, 1992)

A graph G is local tournament orientable if and only if G^+ is bipartite. Moreover, when G^+ is bipartite, for any two vertices $(u, v), (u', v')$ of odd distance in G^+, a local tournament of G must contain exactly one of them as an arc. In particular, the arcs of every local tournament orientation of G correspond to a colour class of G^+. □

orientation completion to a local tournament

Theorem

The orientation completion problem for the class of local tournaments is polynomial time solvable.

Proof: Let $P = (V, A \cup E)$ be a partially oriented graph and let $G = UG(P)$. The arc set A corresponds to a subset S of the vertex set of G^+. According to the Lemma, P can be completed to a local tournament if and only if G^+ is bipartite and S is contained in a colour class of G^+.

Checking whether G^+ is bipartite and in the case when G^+ is bipartite whether S is contained in a colour class of G^+ can be done in polynomial time. $\square$

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The representation extension problem for proper interval graphs

Shorthand for proper interval graph: pig.

Problem (PIG-REPRESENTATION-EXTENSION)

Let G be a proper interval graph and let $\mathcal{I}'$ be a proper interval representation of and induced subgraph H of G. Does there exist a proper interval representation $\mathcal{I}$ of G such that $\mathcal{I}' \subseteq \mathcal{I}$?

Theorem (Klavik, Kratochvil & Vyskocil, 2014)

The problem PIG-REPRESENTATION-EXTENSION *is solvable in polynomial time and we can construct the desired extension in polynomial time when it exists.*

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Proof: We show how to reduce the problem of extending partial proper interval representations of proper interval graphs to the orientation completion problem for the class of acyclic local tournaments.

Suppose that G is a proper interval graph and H is an induced subgraph of G. Given a proper interval representation $I_v, v \in V(H)$ of H, we obtain an orientation of H in such a way that (u, v) is an arc if and only if I_u contains the left endpoint of I_v. The oriented edges together with the remaining edges in G yield a partial (acyclic) orientation of G.

This partial orientation of G can be completed to an acyclic local tournament if and only if the partial representation of H can be extended to a proper interval representation of G. □

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Theorem (Bang-Jensen, Huang & Zhu, 2015)

The orientation completion problem for the class of locally transitive tournaments is $\mathcal{NP}$-complete.

Corollary

It is $\mathcal{NP}$-complete to decide whether a partially oriented proper-circular arc graph has a completion to a locally transitive local tournament.

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Problem (PCA-REPRESENTATION-EXTENSION)

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The PCA-REPRESENTATION-EXTENSION *problem is polynomially solvable and a good extension can be found in polynomial time when it exists.*

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A digraph is an **in-tournament** if the set of in-neighbours of every vertex induces a tournament.

Theorem

The orientation completion problem is polynomial for the class of in-tournaments.

Proposition (Bang-Jensen, Huang & Prisner, 1993)

A graph is chordal if and only if it has an orientation as an acyclic in-tournament.

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What is the complexity of the orientation completion problem for the class of acyclic in-tournaments?

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Thank you very much for your attention!

*

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