The Parameterized Complexity of Finding Paths with Shared Edges

Till Fluschnik, Stefan Kratsch, Rolf Niedermeier, and Manuel Sorge

TU Berlin

October 14, 2015
VIP-Routing

Tegel

Bundestag
VIP-Routing

Tegel

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VIP-Routing

Tegel

Bundestag
The Problem

**Problem:** Minimum Shared Edges (MSE)

**Input:** A simple, undirected graph \( G = (V, E) \), \( s, t \in V \), and two integers \( p \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \).

**Question:** Are there \( p \) s-t paths in \( G \) that share at most \( k \) edges?
The Problem

Are there $p = 3$ $s$-$t$ paths in $G$ that share at most $k = 2$ edges?
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$G$

$s$ $a$ $b$ $c$ $d$ $e$ $t$

$G'$

$s$ $a$ $b$ $c$ $d$ $e$ $t$

$T$

$s$ $a$ $b$ $c$ $d$ $e$ $t$
The Problem

Are there three paths in $G$ that share at most two edges?
The Problem

Are there $p = 3$ paths in $G$ that share at most $k = 2$ edges?

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## Related Work

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[Li et al. ’13]

Aoki et al. [COCOA ’14]
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Theorem

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MSE\((p)\) is FPT.

Theorem

**Minimum Shared Edges** is fixed-parameter tractable with respect to the number \(p\) of paths.
Strategy of Proving “MSE($p$) is FPT”

Instance: $(G, s, t, p, k)$
Strategy of Proving \("\text{MSE}(p)\) is FPT\)
Strategy of Proving “MSE(\(p\)) is FPT”

Instance: \((G, s, t, p, k)\)

Treewidth reduction technique

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The Treewidth Reduction Technique

Theorem (Marx et al. [TALG ’13, Theorem 2.15])

Let \( G \) be a graph, \( T \subseteq V(G) \), and let \( \ell \) be an integer. Let \( C \) be the set of all vertices of \( G \) participating in a minimal \( s\text{-}t \) separator of size at most \( \ell \) for some \( s, t \in T \).

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1. $C \cup T \subseteq V(G^*)$
2. For every $s, t \in T$, a set $L \subseteq V(G^*)$ with $|L| \leq \ell$ is a minimal $s$-$t$ separator of $G^*$ if and only if $L \subseteq C \cup T$ and $L$ is a minimal $s$-$t$ separator of $G$. 
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3. The treewidth of $G^*$ is at most $h(\ell, |T|)$ for some function $h$. 
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4. $G^*[C \cup T]$ is isomorphic to $G[C \cup T]$. 

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The Treewidth Reduction Technique

$T = \{s, t\}, \ell = 2$
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$$T = \{s, t\}, \ell = 2$$
Strategy of Proving “MSE(p) is FPT”

Instance: \((G, s, t, p, k)\)

Treewidth reduction technique with

\(T = \{s, t\} \text{ and } \ell = p - 1.\)
Strategy of Proving “\( \text{MSE}(p) \) is FPT”

**Instance:** \((G, s, t, p, k)\)

1. **Subdivisions**
2. **Treewidth reduction technique** with \(T = \{s, t\}\) and \(\ell = p - 1\).

\[ \text{Instance: } (G^*, s, t, p, k) \]

\[ \text{is solvable in FPT-time wrt. } p \]

\[ \text{is yes-instance if and only if } (G, s, t, p, k) \text{ is yes-instance.} \]
Strategy of Proving “MSE(p) is FPT”

Instance: \((G, s, t, p, k)\)

Subdivisions

\(G \rightarrow H \rightarrow H^* \rightarrow \ldots \rightarrow G^*\)

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Instance: \((G, s, t, p, k)\)

- Subdivisions
  \(G \rightarrow H\)

- Treewidth reduction technique with
  \(T = \{s, t\}\) and
  \(\ell = p - 1\).

- Contractions
  \(H^* \rightarrow G^*\)

---

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From $G$ to $G^*$ (with $\ell = p - 1 = 2$)
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$G$

$s \quad a \quad b \quad t$

$c \quad d \quad e$

$H$

$s \quad a \quad b \quad t$

$c \quad d \quad e$

$H^*$

$s \quad a \quad b \quad t$

$c \quad d \quad e$

Subdivisions

TWRT

$X_{bc}$

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From $G$ to $G^*$ (with $\ell = p - 1 = 2$)
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Instance: $(G, s, t, p, k)$

Subdivisions

Treewidth reduction technique with
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Contractions

$G \xrightarrow{\text{Subdivisions}} H \xrightarrow{} H^* \xrightarrow{} G^*$

Strategy of Proving “MSE($p$) is FPT”

Instance: $(G, s, t, p, k)$

- Subdivisions
- Contractions
- Treewidth reduction technique with $T = \{s, t\}$ and $\ell = p - 1$.

- $\text{tw}(G^*) \leq h(p)$
- 1-to-1 correspondence of all minimal $s$-$t$ cuts of size $\leq p - 1$ of $G$ and $G^*$. 

$G$ \quad $H$ \quad $H^*$ \quad $G^*$
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Instance: $(G^*, s, t, p, k)$
- is solvable in FPT-time wrt. $p$
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Strategy of Proving “MSE(p) is FPT”

Instance: \((G, s, t, p, k)\)

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Subdivisions

\(G\) → \(H\) → \(H^*\) → \(G^*\)

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Conclusion and Remarks

- **MSE** is NP-hard even if maximum degree $\Delta = 5$. 

  Ongoing work indicates that **MSE** remains NP-hard on planar graphs with $\Delta = 4$. Open: $\Delta = 3$?

  **MSE**($k$) is $W[2]$-hard, **MSE**($tw$) is $W[1]$-hard, **MSE**($p$) is FPT. 

  $MSE(p)$ does not admit a polynomial problem kernel (unless $\text{NP} \subseteq \text{coNP}/\text{poly}$).

  Our approach requires the combined use of the treewidth reduction technique and dynamic programming.

  $\Rightarrow$ Running times are of theoretical interest only. Challenge: improve the running time.

- Further research: complexity of **MSE** on special planar graphs, e.g. grids with holes.
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- Further **research**: complexity of **MSE** on *special* planar graphs, e.g. grids with holes.
Thank you.
Yusuke Aoki, Bjarni V. Halldórsson, Magnús M. Halldórsson, Takehiro Ito, Christian Konrad, and Xiao Zhou.
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