Intersection Graphs of Non-crossing Paths

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Abstract:

Intersection representations of graphs are ubiquitous in graph theory and covered in many graph theory textbooks (see, e.g., [2, 4]). For a given graph G, a collection $\mathcal{S}$ of sets, $\{S_v\}_{v \in V(G)}$, is an *intersection representation* of G when $S_u \cap S_v \neq \emptyset$ iff $uv \in E(G)$. Similarly, we say that G is the intersection graph of $\mathcal{S}$.

In this work we consider classes of intersection graphs where the sets are taken from a topological space, are *arc-connected*, and are pairwise *non-crossing*. A set S is *arc-connected* when any two of its points can be connected by a *curve* within the set (note: a *curve* is a homeomorphic image of a closed interval). Notice that when the topological space is a graph, arc-connectedness is precisely the usual connectedness of a graph and curves are precisely paths. Two connected sets S_1, S_2 are said to be *non-crossing* when both $S_1 \setminus S_2$ and $S_2 \setminus S_1$ are arc-connected. The most general case of intersection graphs of non-crossing sets which have been studied are those of non-crossing arc-connected (NC-AC) sets in the plane [3]. These were considered together with the intersection graphs of disks in the plane, or simply *disk* graphs – another non-crossing class. In particular, it has been shown that the recognition of both NC-AC graphs and disk graphs is NP-hard [3].

Our focus is on intersection graphs of non-crossing paths. Note that the class of non-crossing subpaths of a path is precisely the well-studied class of proper interval graphs. We study the graph classes listed below.

- NC paths in a tree.
- NC directed paths in a directed tree.
- NC directed paths in a rooted tree.

These graph classes naturally generalize the algorithmically convenient structure of proper interval graphs. For example, we will also show that the dominating set problem can be solved in polynomial time on the intersection graphs of non-crossing paths in a tree. This contrasts the fact that the dominating set problem is NP-complete on intersection graphs of paths in a tree (i.e., where paths are allowed to cross) [1]. We further characterize all of these classes by finite forbidden induced subgraph characterizations.

References

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